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Stochastic Quantization: A Review

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The present status of the work on the application of the stochastic quantization procedure is reviewed. A compact mathematical introduction to the basic notions of random processes such as Markov processes, Martingales and Fokker-Planck equations is presented. The stochastic quantization procedure is explained in much detail and it is found to possess remarkable features which can not be achieved within the conventional framework of quantum theory. This admits us to give systematic analyses of irreversible quantum dynamics of dissipative systems and the vacuum tunneling phenomena in non-Abelian gauge theory.

PROLOGUE

About ten years ago Nelson (1966, 1967) proposed a probability theoretical framework of quantum mechanics which is now frequently called stochastic quantization procedure. This framework, on the one hand, may have some bearing on academic problems such as hidden variables and the philosophical foundation of quantum theory (Takabayasi, 1977) and, on the other hand, it will find many applications in various fields of quantum physics.

This article is written mainly to give a detailed explanation of the present status of the work on the application of the stochastic quantization procedure and some developments related to it which I have made since I encountered Nelson's book (Nelson, 1967) in 1974.

The stochastic quantization procedure has remarkable features which cannot be obtained within the framework of conventional quantum theory: Firstly it relies on neither Hamiltonian nor Lagrangean but on the equation of motion in the generalized sense. So it seems applicable to the wider class of dynamical systems, that is, not only to nondissipative (conservative) systems but also to dissipative (nonconservative) ones. Secondly one can realize explicitly the behavior of quantized coordinate variables or quantized field variables even when the system is in its energy eigenstate.

Those properties seem adequate in investigating the quantum mechanical description of dynamical systems interacting with chaotic thermal environments, and also the detailed time-dependent description of tunneling phenomena.

In the present article, making full use of the above nice properties of the stochastic quantization, we give systematic analyses of irreversible quantum dynamics of dissipative systems, and the vacuum tunneling phenomena in non-Abelian gauge field theory.

Apology. This is the notes of a lecture on the theory of stochastic quantization which I made at Nagoya University in June 1978. At that time I never thought of publishing the lecture notes in any review article. One year has already passed after the lecture. If I do not recognize Nelson's invited talk at the Einstein Symposium in Berlin in March 1979, it is because these notes were still only in my head. Really, as Nelson mentioned in his talk, the present literature on the theory of stochastic quantization is in a preliminary state. There remain several problems that force us to increase precision from the viewpoint of mathematical physics. To clarify the problems, of course, a wide view on the present status of the theory seems to be needed. Then I believe strongly that it is not meaningless to publish these lecture notes in the scientific literature as a review of the theory of stochastic quantization. I restrict myself to the application of the stochastic quantization because a review of the basic features of it has already appeared in *La Rivista del Nuovo Cimento* recently. In Nelson's words, "It is time, in March 1979, to declare this field of research to be respectable."

1. MATHEMATICAL PRELIMINARIES

In this chapter, prior to the exposition of the stochastic quantization procedure, we give some of the basic notions of random processes such as Wiener process, Markov processes, martingales, and Fokker–Planck equation. We also investigate, with Nelson, the kinematics of random processes.

The main sources for this chapter were Nelson's book (1967), of course, and the texts by Kolmogorov (1933) and Neveu (1970).

1.1. Random Processes, Expectation and Conditional Expectation. By the notion of *random process* X_t, $-\infty < t < \infty$, in $\mathbb{R}^n$ we denote the triplet $(\Omega, \mathfrak{D}(\mathbb{P}\mathrm{rob}), \mathbb{P}\mathrm{rob})$: $\Omega = \prod_{-\infty<t<\infty} \mathbb{R}^n$ is the totality of paths γ in $\mathbb{R}^n$, i.e., γ;

$\mathbb{R} \to \mathbb{R}^n$ and $\mathbb{P}$rob the probability measure defined on a σ algebra $\mathfrak{D}(\mathbb{P}\text{rob})$ of subsets of Ω. Probability theoretically speaking, Ω is a base space or a sample space and the triplet $(\Omega, \mathfrak{D}(\mathbb{P}\text{rob}), \mathbb{P}\text{rob})$ a probability space.

Any random process would be specified by giving the probability measure on Ω; different probability measures define random processes of different natures.

By the notion of *event* ε we denote a condition on the element of Ω such that $\{\omega \in \Omega | \omega \text{ satisfies condition } \varepsilon\}$ belongs to $\mathfrak{D}(\mathbb{P}\text{rob})$. Probability of the event ε is defined to be $\mathbb{P}\text{rob}(\{\omega \in \Omega | \omega \text{ satisfies condition } \varepsilon\})$.

By the notion of *random variable* Z we denote a measurable map from a measurable space $(\Omega, \mathfrak{D}(\mathbb{P}\text{rob}))$ to another one $(S, \mathfrak{B})$, i.e., Z; $\Omega \to S$ such that $Z^{-1}(\mathfrak{B}) \subset \mathfrak{D}(\mathbb{P}\text{rob})$. Expectation or mean value of the random variable Z is defined by

$$\mathbb{E}\{Z\} = \int_{\Omega} Z(\omega) \mathbb{P}\text{rob}(d\omega) \tag{1.1}$$

For example, $X_s(\omega) = \omega_s \in \mathbb{R}^n$, for each $s \in \mathbb{R}$, defines an $\mathbb{R}^n$-valued random variable X_s, where ω_s denotes a cross section of the sample path ω at the time s. The random process X_t, $-\infty < t < \infty$, is equivalent to a family of random variables $(X_s)_{s \in \mathbb{R}}$.

Next we shall introduce a notion of *conditional expectation*. Let $\mathfrak{F}$ be a sub-σ-algebra of $\mathfrak{D}(\mathbb{P}\text{rob})$ and Z a random variable with finite mean, i.e., $Z \in L_1(\Omega, \mathbb{P}\text{rob})$. Then a conditional expectation of Z with respect to $\mathfrak{F}$ is defined to be a $\mathfrak{F}$-measurable function $\mathbb{E}\{Z|\mathfrak{F}\}$ on Ω such that

$$\int_E Z(\omega) \mathbb{P}\text{rob}(d\omega) = \int_E \mathbb{E}\{Z|\mathfrak{F}\} \mathbb{P}\text{rob}(d\omega) \tag{1.2}$$

holds for $\forall E \in \mathfrak{F}$. It is worthwhile to notice that the conditional expectation $\mathbb{E}\{Z|\mathfrak{F}\}$ is nothing but a Radon–Nikodym derivative of a σ-additive function on $\mathfrak{F}$

$$\mu(E) = \int_E Z(\omega) \mathbb{P}\text{rob}(d\omega), \qquad \forall E \in \mathfrak{F} \tag{1.3}$$

with respect to the probability measure $\mathbb{P}$rob. Therefore the conditional expectation, if it exists, is unique with probability one. We prefer to write $\mathbb{E}\{Z|Y\}$ in spite of $\mathbb{E}\{Z|\mathfrak{F}\}$ if the sub-σ-algebra $\mathfrak{F}$ is generated by a random variable Y.

Following are some of the basic properties of the conditional expectation:

(1) If $Z \geqslant 0$, then $\mathbb{E}\{Z|\mathfrak{F}\} \geqslant 0$.

(2) $\mathbb{E}\{aZ + bY|\mathfrak{F}\} = a\mathbb{E}\{Z|\mathfrak{F}\} + b\mathbb{E}\{Y|\mathfrak{F}\}$.

(3) $\mathbb{E}\{\mathbb{E}\{Z|\mathcal{F}\}\}=\mathbb{E}\{Z\}$.
(4) If Z is $\mathcal{F}$ measurable, then $\mathbb{E}\{Z|\mathcal{F}\}=Z$.
(5) If Z is independent of $\mathcal{F}$, then $\mathbb{E}\{Z|\mathcal{F}\}=\mathbb{E}\{Z\}$.
(6) $\mathbb{E}\{\mathbb{E}\{Z|\mathcal{F}\}|\mathcal{G}\}=\mathbb{E}\{Z|\mathcal{G}\}$ for any sub-σ-algebra $\mathcal{G}\subset\mathcal{F}$.

By the notion of *conditional probability* of the event ε with respect to $\mathcal{F}$ we denote a conditional expectation of the characteristic function of a subset $\{\omega\in\Omega|\omega \text{ satisfies } \varepsilon\}$ with respect to $\mathcal{F}$, i.e.,

$$\text{Cond}\{\varepsilon|\mathcal{F}\}=\mathbb{E}\{1_E|\mathcal{F}\} \tag{1.4}$$

where 1_E denotes the characteristic function of $E=\{\omega|\omega \text{ satisfies } \varepsilon\}$. In the simplest case, $\mathcal{F}=\{F,F^c,\phi,\Omega\}$ for a subset $F\in\mathcal{D}(\mathbb{P}\text{rob})$ such that $0<\mathbb{P}\text{rob}(F)<1$, one obtains

$$\text{Cond}\{E|\mathcal{F}\}=\begin{cases}\mathbb{P}\text{rob}(E\cap F)/\mathbb{P}\text{rob}(F) & \text{on } F\\ \mathbb{P}\text{rob}(E\cap F^c)/\mathbb{P}\text{rob}(F^c) & \text{on } F^c\end{cases} \tag{1.5}$$

For the last exposition of this section we prove the following theorem.

Theorem (Bayes).

$$\text{Cond}\{E|F\}=\mathbb{P}\text{rob}(E)\,\text{Cond}\{F|E\}/\mathbb{P}\text{rob}(F) \tag{1.6}$$

Proof. By equation (1.5) we have

$$\begin{aligned}\mathbb{P}\text{rob}(F)\,\text{Cond}\{E|F\}&=\mathbb{P}\text{rob}(E\cap F)\\&=\text{Cond}\{F|E\}\,\mathbb{P}\text{rob}(E).\end{aligned}$$ ■

2. Wiener Process. We shall construct a Wiener process in $\mathbb{R}^n$ as a fundamental example of the random process in what follows. The construction is due to Nelson (1964).

Sample space of the Wiener process is taken to be

$$\Omega=\prod_{-\infty<t<\infty}\dot{\mathbb{R}}^n \tag{1.7}$$

where $\dot{\mathbb{R}}^n$ denotes a one-point compactification of $\mathbb{R}^n$. With the usual product topology, Ω becomes compact and Hausdorff.

Let

$$p^t(x,d^ny)=(4\pi Dt)^{-n/2}\exp\left(-\frac{|x-y|^2}{4DT}\right)d^ny \tag{1.8}$$

$t>0$, be an n-dimensional Gaussian measure centered around $x \in \mathbb{R}^n$, where $D>0$ stands for a diffusion constant.

By the notion of cylinder function on Ω we denote a function f; $\Omega \to \mathbb{R}$ of the form

$$f(\omega) = F(\omega_{t_{-m}}, \ldots, \omega_{t_{-1}}, \omega_{t_1}, \ldots, \omega_{t_m}) \tag{1.9}$$

for $t_{-m} < \cdots < t_{-1} < 0 < t_1 < \cdots < t_m$. Firstly we define the Wiener integral of a cylinder function f by

$$I_x(f) = \int p^{t_{-m+1} - t_{-m}}(x_{-m+1}, dx_{-m}) \cdots p^{-t_{-1}}(x, dx_{-1})$$

$$\times p^{t_1}(x, dx_1) \cdots p^{t_m - t_{m-1}}(x_{m-1}, dx_m) F(x_{-m}, \ldots, x_m) \tag{1.10}$$

For continuous F; $\mathbb{R}^{2m} \to \mathbb{R}$, the integral (1.10) exists because we have

$$I_x(1) = 1 \tag{1.11}$$

Let $C(\Omega)$ be the totality of continuous functions on Ω and $C_{\text{cyl}}(\Omega)$ that of continuous cylinder functions on Ω (a cylinder function f is continuous iff F is continuous). $C(\Omega)$, with the usual supremum norm, is a Banach space. Secondly we define the Wiener integral of a continuous function $g \in C(\Omega)$ as follows.

By the Stone–Weierstrass theorem, we can approximate any function in $C(\Omega)$ uniformly by those in $C_{\text{cyl}}(\Omega)$. The mapping I_x; $f \to I_x(f)$ defines a bounded linear functional on $C_{\text{cyl}}(\Omega)$ of positive type, i.e.,

$$|I_x(f)| \leqslant \|f\| = \sup_{\omega \in \Omega} |f(\omega)| \tag{1.12}$$

$$I_x(f) \geqslant 0 \qquad \text{iff } f \geqslant 0 \tag{1.13}$$

Then, the mapping I_x has a unique extension to a bounded positive linear functional on $C(\Omega)$. The extension should also be denoted I_x.

The Wiener integral of $g \in C(\Omega)$ is defined to be $I_x(g)$. By the Riesz–Kakutani theorem, we find that there exists a regular normalized measure μ_w^x on Ω, indexed by $x \in \mathbb{R}^n$, such that

$$I_x(g) = \int_\Omega g(\omega) \mu_w^x(d\omega) \tag{1.14}$$

holds for $\forall g \in C(\Omega)$. This is the *Wiener measure*.

Adopting the Wiener measure as a probability measure, we can define the Wiener process W_t, $-\infty < t < \infty$, in $\mathbb{R}^n$ by a triplet $(\Omega, \mathfrak{B}(\mu_w^x), \mu_w^x)$.

Concerning the domain of the Wiener measure $\mathfrak{D}(\mu_w^x)$, we have the following theorem.

Theorem [Wiener (Nelson, 1964)]. Let Φ be the totality of continuous paths γ such that $\gamma(t) \in \mathbb{R}^n$ (not $\dot{\mathbb{R}}^n$!) for $-\infty < t < \infty$; then

$$\mu_w^x(\Phi) = 1 \tag{1.15}$$

for x in $\mathbb{R}^n$ (not $\dot{\mathbb{R}}^n$!).

Consequently $\mathfrak{D}(\mu_w^x)$ is taken to be a σ algebra of Borel sets of Φ, i.e., $\mathfrak{B}(\Phi)$.

In this section we have constructed the Wiener measure making a detour along the Stone–Weierstrass–Riesz–Kakutani bypass. This is not the only way; there is a short cut due to Kolmogorov (1933).

We conclude this section with the following two theorems.

Theorem. $u(t,x) = \int u_0(\omega_t)\mu_w^x(d\omega)$ is a solution to the Cauchy problem

$$\frac{\partial}{\partial t} u = D \operatorname{div} \operatorname{grad} u \tag{1.16}$$

$$u(0,x) = u_0(x) \tag{1.17}$$

Theorem. (Feynman–Kac) A solution to the Cauchy problem

$$\frac{\partial}{\partial t} u = D \operatorname{div} \operatorname{grad} u + U(x,t)u \tag{1.18}$$

$$u(0,x) = u_0(x) \tag{1.19}$$

is given by the Wiener integral

$$u(t,x) = \int u_0(\omega_t) \exp\left[\int_0^t U(\omega_s, s) \frac{ds}{2D} \right] \mu_w^x(d\omega) \tag{1.20}$$

1.3. Markov Processes and Time-Reversed Markov Processes. Among various types of random process, the most relevant one to theoretical physics would be Markov processes: *Markov process* in $\mathbb{R}^n X_t$, $-\infty < t < \infty$, is a random process such that

$$\operatorname{Cond}\{X_t \in E \mid X_{t_m} = x_m, \ldots, X_{t_1} = x_1\} = \operatorname{Cond}\{X_t \in E \mid X_{t_m} = x_m\} \tag{1.21}$$

holds with probability one for any time series $-\infty < t_1 < \cdots < t_m < t$, where E is a Borel set of $\mathbb{R}^n$.

By the notion of *transition probability law* of the Markov process we denote

$$P(t, E|s, x) = \mathbb{C}\text{ond}\{X_t \in E | X_s = x\}, \qquad t > s \tag{1.22}$$

which satisfies the following Chapman–Kolmogorov equation:

$$P(t, E|s, x) = \int P(t, E|u, y) P(u, d^n y|s, x) \qquad (t > u > s) \tag{1.23}$$

Next we shall prove that *the time-reversed process of a Markov process is also a Markov process*. The time-reversed process of the Markov process is defined to be a random process X_t^*, $-\infty < t < \infty$, such that

$$X_t^* = X_{-t} \tag{1.24}$$

holds with probability one.

For technical simplicity we replace $\mathbb{R}^n$ by $\mathbb{Z}^n$ ($\mathbb{Z}$ denotes the totality of integers); this corresponds to a lattice approximation.

Theorem. Let X_t, $-\infty < t < \infty$, be a Markov process in $\mathbb{Z}^n$, i.e.,

$$\mathbb{C}\text{ond}\{X_{t_m} = x_m | X_{t_{m-1}} = x_{m-1}, \ldots, X_{t_1} = x_1\} = \mathbb{C}\text{ond}\{X_{t_m} = x_m | X_{t_{m-1}} = x_{m-1}\} \tag{1.25}$$

for any $-\infty < t_1 < \cdots < t_n < \infty$ and $x_1, \ldots, x_m \in \mathbb{Z}^n$. Then we have

$$\mathbb{C}\text{ond}\{X_{t_1} = x_1 | X_{t_2} = x_2, \ldots, X_{t_m} = x_m\} = \mathbb{C}\text{ond}\{X_{t_1} = x_1 | X_{t_2} = x_2\} \tag{1.26}$$

i.e., the time-reversed process X_t^*, $-\infty < t < \infty$, is also Markov.

Proof. The left-hand side of equation (1.26) can be manipulated as

$$\begin{aligned}
&\mathbb{P}\text{rob}\{X_{t_1} = x_1, \ldots, X_{t_m} = x_m\} / \mathbb{P}\text{rob}\{X_{t_2} = x_2, \ldots, X_{t_m} = x_m\} \\
&= \mathbb{C}\text{ond}\{X_{t_m} = x_m | x_{t_{m-1}} = x_{m-1}\} \cdots \mathbb{C}\text{ond}\{X_{t_2} = x_2 | X_{t_1} = x_1\} \mathbb{P}\text{rob}\{X_{t_1} = x_1\} \\
&\quad / \mathbb{C}\text{ond}\{X_{t_m} = x_m | X_{t_{m-1}} = x_{m-1}\} \cdots \mathbb{C}\text{ond}\{X_{t_3} = x_3 | X_{t_2} = x_2\} \mathbb{P}\text{rob}\{X_{t_2} = x_2\} \\
&= \mathbb{C}\text{ond}\{X_{t_2} = x_2 | X_{t_1} = x_1\} \mathbb{P}\text{rob}\{X_{t_1} = x_1\} / \mathbb{P}\text{rob}\{X_{t_2} = x_2\} \\
&= \mathbb{C}\text{ond}\{X_{t_1} = x_1 | X_{t_2} = x_2\}
\end{aligned} \tag{1.27}$$

which is identical with the right-hand side of equation (1.26). ■

Theorem. Let $p(t,x|s,y)$, $t>s$, be a transition probability law of the Markov process X_t, $-\infty<t<\infty$, $\rho(t,x)$ a probability distribution of X_t. Then a transition probability law of the time-reversed Markov process X_t^*, $-\infty<t<\infty, p^*(t,y|s,x), t>s$, is given by

$$p^*(t,y|s,x)=\rho(-t,y)P(-s,x|-t,y)\rho(-s,x)^{-1} \tag{1.28}$$

Proof. A straightforward manipulation yields

$$\begin{aligned} p^*(t,y|s,x) &= \mathbb{C}\text{ond}\{X_t^*=y|X_s^*=x\} \\ &= \mathbb{C}\text{ond}\{X_{-t}=y|X_{-s}=x\} \\ &= \mathbb{P}\text{rob}\{X_{-t}=y,X_{-s}=x\}/\mathbb{P}\text{rob}\{X_{-s}=x\} \\ &= [\mathbb{P}\text{rob}\{X_{-s}=x,X_{-t}=y\}/\mathbb{P}\text{rob}\{X_{-t}=y\}] \\ &\quad \cdot[\mathbb{P}\text{rob}\{X_{-t}=y\}/\mathbb{P}\text{rob}\{X_{-s}=x\}] \\ &= \mathbb{P}\text{rob}\{X_{-t}=y\}\cdot \mathbb{C}\text{ond}\{X_{-s}=x|X_{-t}=y\}/\mathbb{P}\text{rob}\{X_{-s}=x\} \\ &= \rho(-t,y)p(-s,x|-t,y)\rho(-s,x)^{-1} \end{aligned} \tag{1.29}$$

■

1.4 Martingales. Let $\mathcal{A}_t$, $-\infty<t<\infty$, be an increasing family of sub-σ-algebra of $\mathfrak{D}(\mathbb{P}\text{rob})$. By the notion of *martingale* in $\mathbb{R}^n$ with respect to the family $\mathcal{A}_t$, $-\infty<t<\infty$, we denote a random process X_t, $-\infty<t<\infty$, such that X_t is $\mathcal{A}_t$-measurable, belongs to $L_1(\Omega, \mathbb{P}\text{rob})$, and

$$\mathbb{E}\{X_t|\,\mathcal{A}_s\}=X_s, \qquad t>s \tag{1.30}$$

holds with probability one. If the equality in equation (1.30) is replaced by inequalities $\geqslant$ and $\leqslant$, we have *submartingale* and *supermartingale*, respectively.

Following are some of the basic properties of martingales:

(1) if X_t, $-\infty<t<\infty$, is a submartingale (supermartingale), then $m(t) = \mathbb{E}\{X_t\}$ is a increasing (decreasing) function of t.

(2) $m(t)=\text{const}$ iff X_t is a martingale.

1.5 Kinematics of Random Processes. We shall investigate the kinematics of a random process X_t, $-\infty<t<\infty$, in $\mathbb{R}^n$.

It is convenient, following Nelson (1966, 1967), to introduce an increasing family $\mathcal{P}_t$, $-\infty<t<\infty$, and a decreasing family $\mathcal{F}_t$, $-\infty<t<\infty$, of sub-σ-algebras of $\mathfrak{D}$(Prob) such that X_t is $\mathcal{P}_t$ and $\mathcal{F}_t$-measurable. We can always choose such families; e.g., a σ-algebra generated by $\{X_u|u \leqslant t\}$ and that by $\{X_u|u \geqslant t\}$, respectively.

Now we classify various types of random processes from a kinematical point of view in what follows.

Definition. X_t, $-\infty < t < \infty$, is an (S0) process if each X_t belongs to $L_1(\Omega, \mathbb{P}\text{rob})$ and the mapping $t \mapsto X_t$ [from $\mathbb{R}$ to $L_1(\Omega, \mathbb{P}\text{rob})$] is continuous.

Definition. X_t, $-\infty < t < \infty$, is an (S1) process if it is an (S0) process such that

$$DX_t = \lim_{h \downarrow 0} \frac{1}{h} E\{X_{t+h} - X_t | \mathcal{P}_t\} \in L_1(\Omega, \mathbb{P}\text{rob})$$

$$D_* X_t = \lim_{h \downarrow 0} \frac{1}{h} E\{X_t - X_{t-h} | \mathcal{F}_t\} \in L_1(\Omega, \mathbb{P}\text{rob})$$

and the mappings $t \mapsto DX_t, t \mapsto D_* X_t$ are both continuous.

Theorem. Let X_t, $-\infty < t < \infty$, be an (S1) process; then we have

$$\mathbb{E}\{X_b - X_a | \mathcal{P}_a\} = \mathbb{E}\left\{ \int_a^b DX_s \, ds | \mathcal{P}_a \right\} \tag{1.31}$$

$$\mathbb{E}\{X_b - X_a | \mathcal{F}_b\} = \mathbb{E}\left\{ \int_a^b D_* X_s \, ds | \mathcal{F}_b \right\} \tag{1.32}$$

for $a < b$.

The proof of the theorem can be seen in Nelson's book (1967).

Let us define $\mathbb{R} \times \mathbb{R}$-indexed random variables $Y_{(a,b)}$ and $Y^*_{(a,b)}$ through the relations

$$X_b - X_a = \int_a^b DX_s \, ds + Y_{(a,b)} \tag{1.33}$$

$$X_b - X_a = \int_a^b D_* X_s \, ds + Y^*{}_{(a,b)} \tag{1.34}$$

Those are *difference processes*, i.e.,

$$Y^\sharp_{(a,b)} = -Y^\sharp_{(b,a)} \tag{1.35}$$

$$Y^\sharp_{(a,b)} + Y^\sharp_{(b,c)} = Y^\sharp_{(a,c)} \tag{1.36}$$

$Y_{(a,b)}$ is $\mathcal{P}_{\max(a,b)}$ = measurable and $Y^*_{(a,b)} \mathcal{F}_{\min(a,b)}$-measurable. Then there exist two random processes $Y_t, -\infty<t<\infty$, and $Y_t^*, -\infty<t<\infty$, such that $Y_{(a,b)}=Y_b-Y_a$ and $Y^*_{(a,b)}=Y_b^*-Y_a^*$ with probability one.

Theorem. Let $X_t, -\infty<t<\infty$, be an (S1) process; then Y_b-Y_a and $Y_b^*-Y_a^*$ are difference martingales, i.e.,

$$\mathbb{E}\{Y_b-Y_a|\mathcal{P}_a\}=0 \tag{1.37}$$

$$\mathbb{E}\{Y_b^*-Y_a^*|\mathcal{F}_b\}=0 \tag{1.38}$$

for $a<b$.

Definition. $X_t, -\infty<t<\infty$, is an (S2) process if it is an (S1) process such that

$$Y_b-Y_a\in L_2(\Omega,\mathbb{P}\mathrm{rob})$$

$$Y_b^*-Y_a^*\in L_2(\Omega,\mathbb{P}\mathrm{rob})$$

$$\sigma^2(t)=\lim_{h\downarrow 0}\frac{1}{h}\mathbb{E}\{(Y_{t+h}-Y_t)\otimes(Y_{t+h}-Y_t)|\mathcal{P}_t\}\in L_1(\Omega,\mathbb{P}\mathrm{rob})$$

$$\sigma_*^2(t)=\lim_{h\downarrow 0}\frac{1}{h}\mathbb{E}\{(Y_t^*-Y_{t-h}^*)\otimes(Y_t^*-Y_{t-h}^*)|\mathcal{F}_t\}\in L_1(\Omega,\mathbb{P}\mathrm{rob})$$

and the mappings $t\mapsto\sigma^2(t)$ and $t\mapsto\sigma_*^2(t)$ are both continuous.

Theorem. Let $X_t, -\infty<t<\infty$, be an (S2) process; then

$$\mathbb{E}\{(Y_b-Y_a)\otimes(Y_b-Y_a)|\mathcal{P}_a\}=\mathbb{E}\left\{\int_a^b\sigma^2(s)\,ds|\mathcal{P}_a\right\} \tag{1.39}$$

$$\mathbb{E}\{(Y_b^*-Y_a^*)\otimes(Y_b^*-Y_a^*)|\mathcal{F}_b\}=\mathbb{E}\left\{\int_a^b\sigma_*^2(s)\,ds|\mathcal{F}_b\right\} \tag{1.40}$$

for $a<b$.

Definition. $X_t, -\infty<t<\infty$, is an (S3) process if it is an (S2) process such that $\det\sigma^2(t)>0$ and $\det\sigma_*^2(t)>0$.

Nelson clarified the following nice properties of the (S3) process.

Theorem. (Nelson) Let $X_t, -\infty<t<\infty$, be an (S3) process such that the support of $\mathbb{P}$rob is the totality of continuous paths in $\mathbb{R}^n$. Then there exist a Wiener process $W_t, -\infty<t<\infty$, and the time-reversed process $W_t^*, -\infty<t<\infty$, such that W_b-W_a is $\mathcal{P}_{\max(a,b)}$-

measurable and $W_b^* - W_a^* \mathcal{F}_{\min(a,b)}$-measurable, and we have

$$X_b - X_a = \int_a^b DX_s ds + \int_a^b \sigma(s) dW_s \tag{1.41}$$

$$X_b - X_a = \int_a^b D_* X_s ds + \int_a^b \sigma_*(s) dW_s^* \tag{1.42}$$

where the last terms of equations (1.41) and (1.42) are the Itô stochastic integrals.

Proof of the theorem can be seen also in Nelson's book (1967).

We conclude the classification of random processes with three theorems.

Theorem. Let X_t, $-\infty < t < \infty$, be an (S1) process; then

$$\mathbb{E}\{DX_t\} = \mathbb{E}\{D_* X_t\} \tag{1.43}$$

and $X_t = \text{const}$ iff $DX_t = D_* X_t = 0$.

Theorem. If X_t, $-\infty < t < \infty$, is an (S2) process, then

$$\mathbb{E}\{\sigma^2(t)\} = \mathbb{E}\{\sigma_*^2(t)\} \tag{1.44}$$

Theorem. (Nelson) Let X_t, $-\infty < t < \infty$, be an (S1) process and f, g functions defined on $\mathbb{R}^{n+1}$ such that $X_t, Df(X_t, t)$ and $D_* g(X_t, t)$ belong to $L_2(\Omega, \mathbb{P}\text{rob})$, and the mappings $t \mapsto X_t, Df(X_t, t)$ and $D_* g(X_t, t)$ are continuous. Then we have

$$\frac{d}{dt} \mathbb{E}\{f(X_t, t) g(X_t, t)\} = \mathbb{E}\{[DF(X_t, t)] g(X_t, t) + f(X_t, t) D_* g(X_t, t)\} \tag{1.45}$$

Here $Df(X_t, t)$ and $D_* g(X_t, t)$ are the *mean forward derivative and mean backward derivative* defined by

$$Df(X_t, t) = \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}\{f(X_{t+h}, t+h) - f(X_t, t) | \mathcal{P}_t\} \tag{1.46}$$

$$D_* g(X_t, t) = \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}\{g(X_t, t) - g(X_{t-h}, t-h) | \mathcal{F}_t\} \tag{1.47}$$

Proof. We claim that

$$\mathbb{E}\{f(X_b,b)g(X_b,b)-f(X_a,a)g(X_a,a)\}$$
$$=\int_a^b \mathbb{E}\{[Df(X_t,t)]\,g(X_t,t)+f(X_t,t)D_*g(X_t,t)\}\,dt \tag{1.48}$$

which concludes the proof. Equation (1.48) can be verified by dividing the interval $[a,b]$ into n equal parts: $t_j=a+j(b-a)/n$ $(j=0,\ldots,n)$, and passing to the limit $n\to\infty$:

$$\mathbb{E}\{f(X_b,b)g(X_b,b)-f(X_a,a)g(X_a,a)\}$$
$$=\lim_{n\to\infty}\sum_{j=1}^{n-1}\mathbb{E}\{f(X_{t_{j+1}},t_{j+1})g(X_{t_j},t_j)-f(X_{t_j},t_j)g(X_{t_{j-1}},t_{j-1})\}$$
$$=\lim_{n\to\infty}\sum_{j=1}^{n-1}\mathbb{E}\{[f(X_{t_{j+1}},t_{j+1})-f(X_{t_j},t_j)][g(X_{t_j},t_j)+g(X_{t_{j-1}},t_{j-1})]/2$$
$$+[g(X_{t_j},t_j)-g(X_{t_{j-1}},t_{j-1})][f(X_{t_{j+1}},t_{j+1})+f(X_{t_j},t_j)]/2\}$$
$$=\lim_{n\to\infty}\sum_{j=1}^{n-1}\mathbb{E}\{[Df(X_{t_j},t_j)]\,g(X_{t_j},t_j)+f(X_{t_j},t_j)D_*g(X_{t_j},t_j)\}\cdot(b-a)/n$$

= the right-hand side of equation (1.48). ■

Now we proceed to formulating the kinematics of the (S3) process.

Consider an (S3) process $X_t, \infty<t<\infty$, with the following properties satisfied with probability one; $\sigma^2(t)/2=\text{const}=\nu$, $DX_t=b(X_t,t)$ and $D_*X_t=b_*(X_t,t)$, where b and b_* are vector fields of class C^1. In this case one can calculate the mean forward and backward derivatives

$$Df(X_t,t)=\left[\frac{\partial}{\partial t}+b\cdot\text{grad}+\nu\,\text{div}\,\text{grad}\right]f(X_t,t) \tag{1.49}$$

$$D_*g(X_t,t)=\left[\frac{\partial}{\partial t}+b_*\cdot\text{grad}-\frac{\sigma_*^2}{2}\text{grad}\otimes\text{grad}\right]g(X_t,t) \tag{1.50}$$

where we have utilized the Taylor expansions

$$
\begin{aligned}
f(X_{t+h}, t+h) &= f\big(X_t + DX_t \cdot h + (2\nu)^{1/2}(W_{t+h} - W_t), t+h\big) \\
&= f(X_t, t) + \frac{\partial}{\partial t} f(X_t, t) \cdot h \\
&\quad + \operatorname{grad} f(X_t, t) \cdot \big[DX_t \cdot h + (2\nu)^{1/2}(W_{t+h} - W_t) \big] \\
&\quad + \tfrac{1}{2} \operatorname{grad} \otimes \operatorname{grad} f(X_t, t) \cdot 2\nu (W_{t+h} - W_t) \otimes (W_{t+h} - W_t) + o(h)
\end{aligned}
\tag{1.51}
$$

$$
\begin{aligned}
g(X_{t-h}, t-h) &= g(X_t - D_* X_t \cdot h - \sigma_*(t)(W_t^* - W_{t-h}^*), t-h) \\
&= g(X_t, t) - \frac{\partial}{\partial t} g(X_t, t) \cdot h - \operatorname{grad} g(X_t, t) \\
&\quad \cdot \big[D_* X_t \cdot h + \sigma_*(t)(W_t^* - W_{t-h}^*) \big] + \tfrac{1}{2} \operatorname{grad} \otimes \operatorname{grad} g(X_t, t) \\
&\quad \cdot \sigma_*(t)(W_t^* - W_{t-h}^*) \otimes \sigma_*(t)(W_t^* - W_{t-h}^*) + o(h)
\end{aligned}
\tag{1.52}
$$

There are close relations between four quantities ν, $\sigma_*^2/2$, b, and b_*. Namely, we have the following theorem.

Theorem. Suppose f and g belong to $C^2(\mathbb{R}^n) \otimes C_0^1(\mathbb{R})$; then

$$
\int_{-\infty}^{\infty} \mathbb{E}\{ [Df(X_t, t)] \, g(X_t, t) \} \, dt = - \int_{-\infty}^{\infty} \mathbb{E}\{ f(X_t, t) D_* g(X_t, t) \} \, dt
\tag{1.53}
$$

Proof. Integrate equation (1.45). ■

Equation (1.53) can be written in terms of the probability distribution density $\rho(x, t)$ (= Radon–Nikodym derivative of $\mathbb{P}\text{rob}\{X_t \in d^n x\}$ with respect to the Lebesgue measure $d^n x$), obtaining

$$
\int_{\mathbb{R}^{n+1}} [Df(x, t)] \, g(x, t) \rho(x, t) d^n x \, dt = - \int_{\mathbb{R}^{n+1}} f(x, t) \big[D_* g(x, t) \big] \rho(x, t) \, d^n x \, dt
\tag{1.54}
$$

This claims $D^* = -D_*$, where an asterisk superscript means to take an adjoint with respect to the measure $\rho(x,t)\,d^nx\,dt$, i.e.,

$$-\left(\frac{\partial}{\partial t} + b_*\cdot\operatorname{grad} - \frac{\sigma_*^2}{2}\operatorname{grad}\otimes\operatorname{grad}\right)$$
$$= \rho^{-1}\left(-\frac{\partial}{\partial t} - b\cdot\operatorname{grad} - \operatorname{div} b + \nu\operatorname{div}\operatorname{grad}\right)\rho \quad (1.55)$$

By equation (1.55) we find

$$\sigma_*^2(t)/2 = \nu I \qquad (I,\ \text{unit matrix}) \quad (1.56)$$

$$b_*(x,t) = b(x,t) - 2\nu\operatorname{grad}\log\rho(x,t) \quad (1.57)$$

Kinematical quantities we need to describe the kinematics of the process X_t, $-\infty < t < \infty$, are the *mean velocity*, the *osmotic velocity*, and the *mean acceleration*. They are defined, following Nelson (1966, 1967), to be $v(X_t,t) = (DX_t + D_*X_t)/2$, $u(X_t,t) = (DX_t - D_*X_t)/2$ and $a(X_t,t) = (DD_*X_t + D_*DX_t)/2$, respectively. Explicitly we have

$$v = (b + b_*)/2 \quad (1.58)$$

$$u = (b - b_*)/2$$
$$= \nu\operatorname{grad}\log\rho \quad (1.59)$$

$$a = \frac{\partial}{\partial t}v - u\cdot\operatorname{grad}u + v\cdot\operatorname{grad}v - \nu\operatorname{div}\operatorname{grad}u \quad (1.60)$$

The second equality of equation (1.59) is known as the Einstein relation.

In utilizing the forward and backward Fokker–Planck equations

$$\frac{\partial}{\partial t}\rho = -\operatorname{div}(b\rho) + \nu\operatorname{div}\operatorname{grad}\rho \quad (1.61)$$

$$\frac{\partial}{\partial t}\rho = -\operatorname{div}(b_*\rho) - \nu\operatorname{div}\operatorname{grad}\rho \quad (1.62)$$

which will be derived in the next section, one obtains the *equation of continuity*

$$\frac{\partial}{\partial t}\rho = -\operatorname{div}(v\rho) \quad (1.63)$$

Equations (1.59) and (1.63) yield

$$\frac{\partial}{\partial t}u = -\nu\,\mathrm{grad}\,\mathrm{div}\,v - \mathrm{grad}\,u\cdot v \tag{1.64}$$

Finally we find the following basic relations between the three kinematical quantities:

$$\frac{\partial}{\partial t}u(X_t,t) = -\nu\,\mathrm{grad}\,\mathrm{div}\,v(X_t,t) - \mathrm{grad}\,u(X_t,t)\cdot v(X_t,t) \tag{1.65}$$

$$\frac{\partial}{\partial t}v(X_t,t) = a(X_t,t) + u(X_t,t)\cdot\mathrm{grad}\,u(X_t,t) - v(X_t,t)\cdot\mathrm{grad}\,v(X_t,t) + \nu\,\mathrm{div}\,\mathrm{grad}\,u(X_t,t) \tag{1.66}$$

These relations completely specify the kinematics of the process X_t, $-\infty < t < \infty$, provided that the mean acceleration is related to the mean velocity through a certain equation of motion.

1.6. Fokker–Planck Equation. By substituting two functions f and g in equation (1.45) by $f=f(x)$ and $g=1$, respectively, we find

$$\frac{d}{dt}\mathbb{E}\{f(X_t)\} = \mathbb{E}\{Df(X_t)\} = \mathbb{E}\{(b\cdot\mathrm{grad} + \nu\,\mathrm{div}\,\mathrm{grad})f(X_t)\} \tag{1.67}$$

where X_t, $-\infty < t < \infty$, is the same as in the preceding section. Substitution by $f=1$ and $g=g(x)$ also gives

$$\frac{d}{dt}\mathbb{E}\{g(X_t)\} = \mathbb{E}\{D_* g(X_t)\} = \mathbb{E}\{(b_*\cdot\mathrm{grad} - \nu\,\mathrm{div}\,\mathrm{grad})g(X_t)\} \tag{1.68}$$

Equations (1.67) and (1.68) can be written

$$\frac{d}{dt}\int f(x)\rho(x,t)\,d^nx = \int (b\cdot\mathrm{grad} + \nu\,\mathrm{div}\,\mathrm{grad})f(x)\rho(x,t)\,d^nx = \int f(x)(-\mathrm{div}\,b + \nu\,\mathrm{div}\,\mathrm{grad})\rho(x,t)\,d^nx \tag{1.69}$$

$$\frac{d}{dt}\int g(x)\rho(x,t)\,d^nx = \int (b_*\cdot\mathrm{grad} - \nu\,\mathrm{div}\,\mathrm{grad})g(x)\rho(x,t)\,d^nx = \int g(x)(-\mathrm{div}\,b_* - \nu\,\mathrm{div}\,\mathrm{grad})\rho(x,t)\,d^nx \tag{1.70}$$

in terms of the probability distribution density.

Since f and g are arbitrary functions, we have

$$\frac{\partial}{\partial t}\rho = -\operatorname{div}(b\rho) + \nu \operatorname{div}\operatorname{grad}\rho \tag{1.71}$$

$$\frac{\partial}{\partial t}\rho = -\operatorname{div}(b_*\rho) - \nu \operatorname{div}\operatorname{grad}\rho \tag{1.72}$$

These are the *forward Fokker–Planck equation* and the *backward Fokker–Planck equation*, respectively.

2. STOCHASTIC QUANTIZATION

This chapter is devoted to an exposition of the stochastic quantization procedure in both cases of quantum mechanics and quantum field theory.

The main sources for this chapter were Nelson's book (1967), as usual, and the present author's paper (Yasue, 1978a).

2.1 Quantum Mechanics. Let us consider a classical dynamical system consists of n configuration variables $q(t) = (q^1(t), \ldots, q^n(t))$. They satisfy *Newton's equation of motion*

$$m\ddot{q}(t) = e\left[-\frac{\partial}{\partial t}A(q(t),t) - \operatorname{grad}V(q(t),t)\right] + eF(q(t),t)\cdot\dot{q}(t) \tag{2.1}$$

where m and e are mass and charge parameters, $A(q,t)$ and $V(q,t)$ vector and scalar potentials for external electromagnetic fields, and

$$\begin{aligned} F(q,t) &= \operatorname{grad} \wedge A(q,t) \\ &= \operatorname{alt}\operatorname{grad} \otimes A(q,t) \end{aligned} \tag{2.2}$$

field strength tensor.

To quantize such a dynamical system, the stochastic quantization procedure demands the quantized configuration variable to be a random process X_t, $-\infty < t < \infty$, on the configuration space $\mathbb{R}^n$. It is the (S3) process considered in Section 5 such that *ν is taken to be $\hbar/2m$, where $\hbar$ denotes Planck's constant divided by 2π.*

As we have shown at the end of Section 5, the kinematics of the quantized configuration variable X_t, $-\infty < t < \infty$, is subjected to the relations

$$\frac{\partial}{\partial t}u(X_t,t) = -\frac{\hbar}{2m}\operatorname{grad}\operatorname{div}v(X_t,t) - \operatorname{grad}u(X_t,t)\cdot v(X_t,t) \tag{2.3}$$

$$\begin{aligned} \frac{\partial}{\partial t}v(X_t,t) = {} & a(X_t,t) + u(X_t,t)\cdot\operatorname{grad}u(X_t,t) - v(X_t,t)\cdot\operatorname{grad}v(X_t,t) \\ & + \frac{\hbar}{2m}\operatorname{div}\operatorname{grad}u(X_t,t) \end{aligned} \tag{2.4}$$

To make the interrelation between three kinematical quantities $v(X_t, t)$, $u(X_t, t)$, and $a(X_t, t)$ closed, we assume with Nelson (1966, 1967) *Newton's equation of motion in terms of the mean acceleration and the mean velocity*

$$ma(X_t, t) = e\left[-\frac{\partial}{\partial t}A(X_t, t) - \operatorname{grad} V(X_t, t)\right] + eF(X_t, t)\cdot v(X_t, t) \quad (2.5)$$

Kinematics of the quantized configuration variable X_t, $-\infty < t < \infty$, is completely specified by solving equations (2.3), (2.4), and (2.5).

This is done by assuming the integrability of the *mean momentum*, i.e., we demand

$$mv(X_t, t) + eA(X_t, t) = \hbar \operatorname{grad} S(X_t, t) \quad (2.6)$$

with probability one, where S belongs to $C^2(\mathbb{R}^n)\otimes C^1(\mathbb{R})$. Since the integrability of the osmotic velocity has been already verified, obtaining

$$u(X_t, t) = \frac{\hbar}{m} \operatorname{grad} R(X_t, t) \quad (2.7)$$

with $R = \log(\rho)^{1/2}$ [see equation (1.59)], one can convert equations (2.3), (2.4), and (2.5) into the following two equations for two unknown kinematical quantities S and R:

$$\frac{\partial}{\partial t}S = \frac{\hbar}{2m}\left(\operatorname{div}\operatorname{grad} R + |\operatorname{grad} R|^2 - |\operatorname{grad} S|^2\right)$$

$$+ \frac{e}{m}A\cdot\operatorname{grad} S - \frac{e^2}{2m\hbar}A^2 - \frac{e}{\hbar}V + \text{const of integration} \quad (2.8)$$

$$\frac{\partial}{\partial t}R = -\frac{\hbar}{2m}\operatorname{div}\operatorname{grad} S - \frac{\hbar}{m}\operatorname{grad} R\cdot\operatorname{grad} S$$

$$+ \frac{e}{m}A\cdot\operatorname{grad} R + \frac{e}{2m}\operatorname{div} A \quad (2.9)$$

Note that the constant of integration in equation (2.8) can be made zero by a suitable choice of S.

We introduce the wave function or the probability *amplitude of the system* $\psi(x, t) = \exp[R(x, t) + iS(x, t)]$ $(i^2 = -1)$. This is only an unknown kinematical quantity of the quantized configuration variable X_t, $-\infty < t < \infty$, which is, by equations (2.8) and (2.9), subjected to an equation

$$i\hbar\frac{\partial}{\partial t}\psi = \left(\frac{1}{2m}|-i\hbar\operatorname{grad} - eA|^2 + V\right)\psi \quad (2.10)$$

This is nothing but *the Schrödinger equation.*

Finally we find that *the kinematics of the quantized configuration variable is completely specified by solving the Schrödinger equation* (2.10). The probabilistic interpretation of the wave function evidently holds, because we have

$$\begin{aligned} |\psi(x,t)|^2 d^n x &= \exp\left[2R(x,t)\right] d^n x \\ &= \rho(x,t)\, d^n x \\ &= \mathbb{P}\mathrm{rob}\{X_t \in d^n x\} \end{aligned} \tag{2.11}$$

2.2 Quantum Field Theory. In this section we investigate the stochastic quantization of wave fields. We make use of the basic notions of the nonstandard analysis (Davis, 1977; Nelson, 1976) for the purpose of treating infinitely many degrees of freedom consistently.

First we shall develop a theory of infinite-dimensional random processes.

We fix a free ultrafilter (Davis, 1977) $\mathbb{F}$ on $\mathbb{N}$. Let $^*\mathbb{E} = \Pi_{n\in\mathbb{N}}\mathbb{R}^n/\mathbb{F}$ be the ultra-Euclidean space, i.e., a quotient space of $\Pi_{n\in\mathbb{N}}\mathbb{R}^n$ with respect to the equivalence relation $\sim$

$$a \sim b \Leftrightarrow \{n \in \mathbb{N} \mid a^{(n)} = b^{(n)}\} \in \mathbb{F} \tag{2.12}$$

where $a = \{a^{(n)}\}_{n\in\mathbb{N}} = \{(a_1^{(n)}, \ldots, a_n^{(n)})\}_{n\in\mathbb{N}}$ and $b = \{b^{(n)}\}_{n\in\mathbb{N}} = \{(b_1^{(n)}, \ldots, b_n^{(n)})\}_{n\in\mathbb{N}}$ belong to $\Pi_{n\in\mathbb{N}}\mathbb{R}^n$. The equivalence class which contains $a = \{a^{(n)}\}_{n\in\mathbb{N}}$ should be denoted by $^*[a^{(n)}] \in {}^*\mathbb{E}$.

The ultra-Euclidean space $^*\mathbb{E}$ possesses a structure of linear space over the ultrareal field $^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/\mathbb{F}$, and that of Euclidean space with respect to the inner product

$$\begin{aligned} \langle [a^{(n)}], [b^{(n)}] \rangle &= {}^*[\langle a^{(n)}, b^{(n)} \rangle] \\ &= {}^*\left[\sum_{p\leq n} a_p^{(n)} b_p^{(n)}\right] \in {}^*\mathbb{R} \end{aligned} \tag{2.13}$$

By the notion of random process X_t, $-\infty < t < \infty$, in $^*\mathbb{E}$ (i.e., an infinite-dimensional random process) we denote a triplet $({}^*\Omega, \mathfrak{B}({}^*\mathbb{P}\mathrm{rob}), {}^*\mathbb{P}\mathrm{rob})$; $^*\Omega$ is a totality of paths γ in $^*\mathbb{E}$ and $^*\mathbb{P}\mathrm{rob}$ a nonstandard probability measure on $^*\Omega$, i.e., a σ-additive set function from the σ algebra $\mathfrak{B}({}^*\mathbb{P}\mathrm{rob})$ to $^*[0,1] = [0,1]^{\mathbb{N}}/\mathbb{F}$ such that $^*\mathbb{P}\mathrm{rob}(\phi) = 0$ and $^*\mathbb{P}\mathrm{rob}({}^*\Omega) = 1$. It is worthwhile to notice that we have generalized the concept of probability, in the sense of the nonstandard analysis, to allow its infinitesimal values.

Secondly we generalize the concept of real-valued functions defined on a D-dimensional Euclidean space $\mathbb{R}^D$.

Let $\mathbb{S}(\mathbb{R}^D)$ be the Schwartz space over $\mathbb{R}^D$ and $\{e_p\}_{p\in\mathbb{N}}\subset\mathbb{S}(\mathbb{R}^D)$ a complete normalized orthogonal system C.N.O.S. in $L^2(\mathbb{R}^D)$. Then one can associate a unique $^*\mathbb{R}$-valued function ϕ on $\mathbb{R}^D$ with each element $^*[a^{(n)}]$ in $^*\mathbb{E}$ as follows:

$$\begin{aligned}\phi(\cdot) &= {}^*[\phi_n(\cdot)] \\ &= {}^*\left[\sum_{p\leqslant n} a_p^{(n)} e_p(\cdot)\right]\end{aligned} \tag{2.14}$$

Totality of such $^*\mathbb{R}$-valued functions is denoted by $^*\mathfrak{S}(\mathbb{R}^D)$ and called "Kawabata space" (Kurata, 1977) over $\mathbb{R}^D$. It is homeomorphic to $^*\mathbb{E}$ if we define an inner product

$$\begin{aligned}\langle\phi,\chi\rangle &= \left\langle {}^*\left[\sum_{p\leqslant n} a_p^{(n)} e_p\right], {}^*\left[\sum_{q\leqslant n} b_q^{(n)} e_q\right]\right\rangle \\ &= {}^*\left[\sum_{p,q\leqslant n} a_p^{(n)} b_q^{(n)} \langle e_p, e_q\rangle\right] \\ &= {}^*\left[\sum_{p\leqslant n} a_p^{(n)} b^{(n)}{}_p\right] \\ &= \langle {}^*[a^{(n)}], {}^*[b^{(n)}]\rangle \\ &= \langle a, b\rangle\end{aligned} \tag{2.15}$$

for any two elements $\phi={}^*[\Sigma_{p\leqslant n} a_p^{(n)} e_p]$ and $\chi={}^*[\Sigma_{p\leqslant n} b_p^{(n)} e_p]$ in $^*\mathfrak{S}(\mathbb{R}^D)$.

Each function $\phi\in{}^*\mathfrak{S}(\mathbb{R}^D)$ is locally differentiable and integrable in the sense that

$$\operatorname{grad}\phi(x) = {}^*\left[\sum_{p\leqslant n} a_p^{(n)} \operatorname{grad} e_p(x)\right] \in {}^*\mathbb{R} \tag{2.16}$$

and

$$\int_{\mathbb{R}^D} \phi(x)\, d^D x = {}^*\left[\sum_{p\leqslant n} a_p^{(n)} \int_{\mathbb{R}^D} e_p(x)\, d^D x\right] \in {}^*\mathbb{R} \tag{2.17}$$

hold. Local product and tensor product of ϕ and χ can be defined to be

$$\phi(x)\chi(x) = {}^*\left[\sum_{p,q\leqslant n} a_p^{(n)} b_q^{(n)} e_p(x) e_q(x)\right] \in {}^*\mathbb{R} \tag{2.18}$$

and

$$\phi \otimes \chi = {}^*\left[\sum_{p,q \leq n} a_p^{(n)} b_q^{(n)} e_p \otimes e_q \right]$$
$$\in {}^*\mathfrak{S}(\mathbb{R}^D) \otimes {}^*\mathfrak{S}(\mathbb{R}^D) \tag{2.19}$$

respectively.

Let $X^{(n)}(t), -\infty < t < \infty$, be an (S3) process in $\mathbb{R}^n$ such that the probability measure is concentrated on the totality of continuous paths and the following properties are satisfied with probability one: $\sigma^2(t)/2 = \text{const} = \nu$, $DX^{(n)}(t) = b^{(n)}(X^{(n)}(t), t)$ and $D_* X^{(n)}(t) = b_*^{(n)}(X^{(n)}(t), t)$, where $b^{(n)}$ and $b_*^{(n)}$ are vector fields of class C^1. Starting with a family of such random processes $\{X^{(n)}(t)\}_{n \in \mathbb{N}}$, one can construct a random process $\Psi_t, -\infty < t < \infty$, in ${}^*\mathfrak{S}(\mathbb{R}^D)$ by

$$\Psi_t = {}^*\left[\sum_{p \leq n} X_p^{(n)}(t) e_p \right] \tag{2.20}$$

The triplet $({}^*\Omega, \mathfrak{D}({}^*\mathbb{P}\mathrm{rob}), {}^*\mathbb{P}\mathrm{rob})$ is defined to be ${}^*\Omega = \prod_{n \in \mathbb{N}} \Omega^{(n)}/\mathbb{F}$, $\mathfrak{D}({}^*\mathbb{P}\mathrm{rob}) = \prod_{n \in \mathbb{N}} \mathfrak{D}(\mathbb{P}\mathrm{rob}^{(n)})/\mathbb{F}$ and ${}^*\mathbb{P}\mathrm{rob} = {}^*[\mathbb{P}\mathrm{rob}^{(n)}]$, where the triplet $(\Omega^{(n)}, \mathfrak{D}(\mathbb{P}\mathrm{rob}^{(n)}), \mathbb{P}\mathrm{rob}^{(n)})$ denotes the probability space of the process $X^{(n)}(t), -\infty < t < \infty$.

Thirdly we shall introduce, with Kawabata and Kurata (1977), the functional derivative and the functional integral of a ${}^*\mathbb{R}$-valued functional on ${}^*\mathfrak{S}(\mathbb{R}^D)$ of the type $F(\psi) = {}^*[F_n(x^{(n)})]$:

$$\frac{\delta F}{\delta \psi} = {}^*\left[\sum_{p < n} \frac{\partial F_n(x^{(n)})}{\partial x_p^{(n)}} e_p \right] \in {}^*\mathfrak{S}(\mathbb{R}^D) \tag{2.21}$$

$$\int F(\psi) \delta\psi = {}^*\left[\int F_n(x^{(n)}) d^n x^{(n)} \right] \in {}^*\mathbb{R} \tag{2.22}$$

where $\psi = {}^*[\sum_{p \leq n} x_p^{(n)} e_p]$. In terms of the functional integral, a nonstandard probability distribution of Ψ_t is given

$$\begin{aligned} {}^*\mathbb{P}\mathrm{rob}\{\Psi_t \in \delta\Psi\} &= {}^*\{\mathbb{P}\mathrm{rob}^{(n)}[X^{(n)}(t) \in d^n x^{(n)}]\} \\ &= {}^*[\rho_n(x^{(n)}, t) d^n x^{(n)}] \\ &= {}^*[\rho_n(x^{(n)}, t)] \cdot {}^*[d^n x^{(n)}] \\ &= P(\psi, t) \delta\psi \end{aligned} \tag{2.23}$$

where $P(\psi, t) = {}^*[\rho_n(x^{(n)}, t)]$ and $\delta\psi = {}^*[d^n x^{(n)}]$.

The nonstandard probability distribution density P satisfies infinite-dimensional versions of Fokker–Planck equations (1.71) and (1.72):

$$\frac{\partial}{\partial t}P(\psi,t) = -\int d^Dx \frac{\delta}{\delta\psi(x)}\{[V_t\psi(x)]P(\psi,t)\} + \nu\int d^Dx \frac{\delta^2}{\delta\psi(x)^2}P(\psi,t) \tag{2.24}$$

$$\frac{\partial}{\partial t}P(\psi,t) = -\int d^Dx \frac{\delta}{\delta\psi(x)}\{[U_t\psi(x)]P(\psi,t)\} - \nu\int d^Dx \frac{\delta^2}{\delta\psi(x)^2}P(\psi,t) \tag{2.25}$$

where V_t and U_t denote transformations

$$V_t;\ \psi \mapsto V_t\psi(x) = {}^*\left[\sum_{p<n} b_p^{(n)}(x^{(n)},t)e_p\right] \tag{2.26}$$

$$U_t;\ \psi \mapsto U_t\psi(x) = {}^*\left[\sum_{p<n} b_{*p}^{(n)}(x^{(n)},t)e_p\right] \tag{2.27}$$

Similarly the mean forward and backward derivatives induced by the process Ψ_t are

$$\begin{aligned} DF(\Psi_t,t) &= \lim_{h\downarrow 0}\frac{1}{h}\mathbb{E}\{F(\Psi_{t+h},t+h) - F(\Psi_t,t)|{}^*\mathcal{P}_t\} \\ &= \left[\frac{\partial}{\partial t} + \int d^Dx V_t\psi(x)\frac{\delta}{\delta\psi(x)} + \nu\int d^Dx\frac{\delta^2}{\delta\psi(x)^2}\right]F(\Psi_t,t) \end{aligned} \tag{2.28}$$

$$\begin{aligned} D_*F(\Psi_t,t) &= \lim_{h\downarrow 0}\frac{1}{h}\mathbb{E}\{F(\Psi_t,t) - F(\Psi_{t-h},t-h)|{}^*\mathcal{F}_t\} \\ &= \left[\frac{\partial}{\partial t} + \int d^Dx U_t\psi(x)\frac{\delta}{\delta\psi(x)} - \nu\int d^Dx\frac{\delta^2}{\delta\psi(x)^2}\right]F(\Psi_t,t) \end{aligned} \tag{2.29}$$

where ${}^*\mathcal{P}_t$, $-\infty < t < \infty$, is an increasing family of sub-σ-algebra of $\mathfrak{D}({}^*\mathbb{P}\mathrm{rob})$ and ${}^*\mathcal{F}_t$, $-\infty < t < \infty$, a decreasing one.

Finally we can proceed with the stochastic quantization of wave fields. Let us consider a real scalar field $\psi(x,t)$ on $\mathbb{R}^D$ with field equation

$$\ddot{\psi}(x,t) = -\frac{\delta}{\delta\psi(x,t)}\left[\frac{1}{2}\int(|\mathrm{grad}\,\psi|^2 + m^2\psi^2)d^Dy + J(\psi)\right] \tag{2.30}$$

where m is a parameter with dimension L^{-1} and $J(\psi) = {}^*[J_n(x^{(n)})]$ the interaction potential.

The stochastic quantization procedure demands *the quantized field variable to be a random process* Ψ_t, $-\infty < t < \infty$, *in* ${}^*\mathfrak{S}(\mathbb{R}^D)$ *of the aforementioned type with* $\nu = \hbar/2$. Kinematics of the quantized field Ψ_t is completely specified by the following *field equation in the generalized sense*:

$$\tfrac{1}{2}(DD_* + D_*D)\Psi_t = -\frac{\delta}{\delta\Psi_t}\left[\frac{1}{2}\int(|\mathrm{grad}\,\Psi_t|^2 + m^2\Psi_t^2)d^Dy + J(\Psi_t)\right] \tag{2.31}$$

This can be manipulated as

$$\frac{1}{2}\left[\frac{\partial}{\partial t}U_t\psi + \int d^Dy\,V_t\psi(y)\frac{\delta}{\delta\psi(y)}U_t\psi + \frac{\hbar^2}{2}\int d^Dy\frac{\delta^2}{\delta\psi(y)^2}U_t\psi + \frac{\partial}{\partial t}V_t\psi\right.$$
$$\left.\left. + \int d^Dy\,U_t\psi(y)\frac{\delta}{\delta\psi(y)}V_t\psi - \frac{\hbar^2}{2}\int d^Dy\frac{\delta^2}{\delta\psi(y)^2}V_t\psi\right]\right|_{\psi=\Psi_t}$$
$$= -\frac{\delta}{\delta\psi}\left[\frac{1}{2}\int(|\mathrm{grad}\,\psi|^2 + m^2\psi^2)d^Dy + J(\psi)\right]\Bigg|_{\psi=\Psi_t} \tag{2.32}$$

Next we make an additional assumption on the transformations V_t and U_t:

$$\tfrac{1}{2}(V_t + U_t)\psi = \hbar\frac{\delta}{\delta\psi}S(\psi,t) \tag{2.33}$$

where $S(\cdot,t)$ is a functional on ${}^*\mathfrak{S}(\mathbb{R}^D)$ of the type $S(\psi,t) = {}^*[S_n(x^{(n)},t)]$.

Let us introduce a wave functional

$$\Omega(\psi,t) = [P(\psi,t)]^{1/2}\exp[iS(\psi,t)] \tag{2.34}$$

on ${}^*\mathfrak{S}(\mathbb{R}^D)$. Then equations (2.24), (2.25), (2.30), and (2.33) yield the

Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Omega(\psi,t)=\frac{1}{2}\int d^Dx\left[-\hbar^2\frac{\delta^2}{\delta\psi(x)^2}+|\mathrm{grad}\,\psi(x)|^2+m^2\psi(x)^2\right]$$

$$\times\,\Omega(\psi,t)+J(\psi)\Omega(\psi,t) \tag{2.35}$$

Kinematics of the quantized field Ψ_t, $-\infty<t<\infty$, would be determined completely by solving a Cauchy problem: equation (2.35) with an initial condition $\Omega(\psi,0)=\Omega_0(\psi)$, say.

Thus the stochastic quantization is shown to provide the same representation as the canonical one:

$$i\hbar\frac{\partial}{\partial t}\Omega(\psi,t)=\frac{1}{2}\int d^Dx\left[\pi(x)^2+|\mathrm{grad}\,\psi(x)|^2+m^2\psi(x)^2\right]$$

$$\times\,\Omega(\psi,t)+J(\psi)\Omega(\psi,t)$$

$$\left[\psi(x),\pi(x')\right]=i\hbar\delta^D(x-x') \tag{2.36}$$

3. IRREVERSIBLE QUANTUM DYNAMICS

In this chapter we present one of the most significant applications of the stochastic quantization procedure; irreversible quantum dynamics of open dynamical systems interacting with chaotic thermal environments.

The main source for this chapter was the present author's paper (Yasue, 1978b).

3.1 Schrödinger–Langevin Equation. Quantum mechanics has been developed to deal with closed or isolated dynamical systems. There a time evolution of such an isolated system is assumed to be a one-parameter unitary group on a Hilbert space. Its infinitesimal generator is the Hamiltonian.

How can one treat open dynamical systems interacting with the external world within the realm of quantum mechanics? Let us try to formulate the problem as faithfully as possible. When we intend to realize a quantum mechanical time evolution of such an open system, we have to start with the time evolution of the total system, i.e., the system plus the external world, generated by the total Hamiltonian

$$H_T=H_S\otimes I+I\otimes H_E+H_I \tag{3.1}$$

where $\otimes$ denotes the tensor product, H_S the Hamiltonian of the system without the interaction with the external world, H_E that of the external world without the interaction with the system, and H_I represents that between the system and the external world (Davies, 1976). Then we encounter two difficulties: Firstly, even when the system itself is a simple dynamical system with finite number of degrees of freedom, we have to deal with complicated ones with infinitely many degrees of freedom including the external world. Secondly it is difficult to specify rigorously the interaction between the system and the external world in each case.

To avoid such difficulties inherent in the fundamental approach, one may adopt a rather phenomenological approach incorporating phenomenological irreversible properties such as dissipations and fluctuations into quantum mechanics.

We consider the external world, with which an open dynamical system interacts, as a chaotic thermal environment (a heat reservoir). It is to avoid the difficulty in realizing the complicated interaction between the system and the external world. Namely, the influence of the external world on the system is assumed to be purely statistical in nature.

In classical mechanics, irreversible dynamics of such an open system is described by the so-called Langevin equation

$$m\ddot{x}(t) = -\beta\dot{x}(t) - \operatorname{grad} V(x(t),t) + A(t) \tag{3.2}$$

where $x(t)$ denotes configuration variable of the open system, $V(x,t)$ a usual potential function, m a mass parameter, and $A(t)$ a Gaussian white noise (Hida, 1975a; Hida and Hitsuda, 1976) with mean 0 and variance

$$\mathbb{E}\{A(t)\otimes A(u)\} = 2D\delta(t-u) \tag{3.3}$$

Note that the complicated interaction between the system and the thermal environment is characterized by the friction coefficient β and the diffusion constant

$$D = \beta k_B T \tag{3.4}$$

where T stands for a temperature of the thermal environment and k_B is Boltzmann's constant (Chandrasekhar, 1943).

Now we shall derive a quantum mechanical version of the Langevin equation (3.2) which might well describe a quantum mechanical behavior of the open system.

Here we have to notice that neither the conventional canonical quantization procedure nor the path integral one would be applied to the present case because of the absence of the Hamiltonian or the Lagrangean for the open system described by the Langevin equation (3.2). Therefore we are obliged to make use of a different method, in which neither Hamiltonian nor Lagrangean is needed. *This is an emergence of the stochastic quantization procedure.*

The stochastic quantization of the open system (3.2) can be performed straightforwardly. All the procedures are the same as in Section 7, provided that Newton's equation of motion in terms of the mean acceleration and the mean velocity (2.5) is replaced by the following Langevin equation in terms of them:

$$ma(X_t,t) = -\beta v(X_t,t) - \operatorname{grad} V(X_t,t) + A(t) \tag{3.5}$$

Correspondingly the Schrödinger equation (2.10) becomes

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\psi(x,t) &= \left[-\frac{\hbar^2}{2m}\operatorname{div}\operatorname{grad} + V(x,t) - x\cdot A(t) + \frac{\beta}{m}\hbar S(x,t) \right]\psi(x,t) \\ &= \left[-\frac{\hbar^2}{2m}\operatorname{div}\operatorname{grad} + V(x,t) - x\cdot A(t) + \frac{i\beta}{2m}\hbar\log\frac{\bar{\psi}(x,t)}{\psi(x,t)} \right]\psi(x,t) \end{aligned} \tag{3.6}$$

The random potential $-x\cdot A(t)$ in equation (3.6) may be replaced by a more general one $R(x,t)$. Notice that the probabilistic interpretation (2.11) still holds even though equation (3.6) is no longer linear.

Equation (3.6) was first derived heuristically by Kostin (1972, 1975), later on by Razavy (1977) in utilizing Schrödinger's quantization procedure via variational problem, and has been called the "*Schrödinger–Langevin equation*." The derivation explained above is based on the work by the present author (Yasue, 1976, 1977a) and by Skagerstam (1977). The dissipative and irreversible characters of the Schrödinger – Langevin equation (3.6) have been investigated by many authors (for example, see Messer 1979).

3.2 Josephson Effect with Thermal Fluctuation. Many of the nonequilibrium phenomena characteristic of open dynamical systems interacting with chaotic thermal environments were described classically by the Langevin equation (3.2). So it seems adequate to make use of the Schrödinger–Langevin equation (3.6) in the quantum mechanical analysis of such nonequilibrium phenomena.

As one of the interesting examples let us investigate the thermal decay of the AC Josephson current near the critical temperature T_c.

It is well known that in the ground state of the total system each two electrons at the Fermi surface in superconducting media form bound pairs (Cooper pairs) and behave as Bose particles. At the absolutely zero temperature all the electrons in the superconducting media degenerate into the ground state because the pairs obey Bose statistics. Then there appear no thermal agitations and the collective motion of Cooper pairs cause the superconducting current. Such a collective motion can be described by the following macroscopic Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t)=\left[-\frac{\hbar^2}{2m}\Delta+V(\mathbf{r},t)\right]\psi(\mathbf{r},t) \tag{3.7}$$

where Δ denotes the three-dimensional Laplacian and m the effective mass of the bound pair. Note that the absolute square of the macroscopic wave function $\rho(\mathbf{r},t)=|\psi(\mathbf{r},t)|^2$ coincides with the number density of the pairs in the ground state.

At a finite temperature $T\simeq T_c$ a number of pairs proportional to the Boltzmann factor $\exp(-E_b/k_B T)$ are broken into normal electrons by the complicated interaction due to the thermal environment, where E_b denotes the bound energy of each pair. Those normal electrons produce complications in the collective motion of the remaining Cooper pairs and then these pairs suffer dissipations and fluctuations by the thermal motion of normal electrons. We shall refer the macroscopic Schrödinger–Langevin equation

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t)=\left[-\frac{\hbar^2}{2m}\Delta+V(\mathbf{r},t)+R(\mathbf{r},t)+\frac{\gamma\hbar}{2i}\log\frac{\bar{\psi}(\mathbf{r},t)}{\psi(\mathbf{r},t)}\right]\psi(\mathbf{r},t)$$

$$\left(\gamma=\frac{\beta}{m}\right) \tag{3.8}$$

to describe phenomenologically such a collective motion of the Cooper pairs near the critical temperature T_c.

Let us consider the collective motion of Cooper pairs at the finite temperature $T\simeq T_c$ across a Josephson junction which consists of two superconductors connected by a thin layer of insulator. As we impose a macroscopic AC electrostatic potential difference $V(t)$ across the junction, the potential function $V(\mathbf{r},t)$ in equation (3.8) should be uniform in each superconducting region (say regions 1 and 2), that is,

$$V(\mathbf{r},t)=\begin{cases} qV_1(t) & \text{if } \mathbf{r}\in\text{region 1} \\ qV_2(t) & \text{if } \mathbf{r}\in\text{region 2} \end{cases} \tag{3.9}$$

with $V_1(t)-V_2(t)=V(t)$, where q denotes the effective charge of each pair. Moreover we may be allowed to assume that the random potential $R(\mathbf{r},t)$ in equation (3.8) be uniform in each superconducting region because the junction is small enough compared with the thermal environment. Namely, we have

$$R(\mathbf{r},t)=\begin{cases} R_1(t) & \text{if } \mathbf{r}\in\text{region 1} \\ R_2(t) & \text{if } \mathbf{r}\in\text{region 2,} \end{cases} \tag{3.10}$$

where $R(t)$'s are two independent Gaussian white noises such that $R(t)=R_1(t)-R_2(t)$ is a Gaussian white noise with mean 0 and variance

$$\mathbb{E}\{R(t)R(u)\}=2G\delta(t-u) \tag{3.11}$$

Note that the diffusion constant G of the white noise $R(t)$ is characteristic of the special choice of the junction and the temperature of the environment.

Under those assumptions the macroscopic wave function $\psi(\mathbf{r},t)$ in equation (3.8) can be approximated as

$$\psi(\mathbf{r},t)=\begin{cases} \psi_1(t) & \text{if } \mathbf{r}\in\text{region 1} \\ \psi_2(t) & \text{if } \mathbf{r}\in\text{region 2} \end{cases} \tag{3.12}$$

and equation (3.8) reduces to the following coupled nonlinear ordinary differential equations:

$$i\hbar\dot{\psi}_1(t)=[qV_1(t)+R_1(t)+\gamma\hbar\theta_1(t)]\psi_1(t)+K\psi_2(t) \tag{3.13}$$

$$i\hbar\dot{\psi}_2(t)=[qV_2(t)+R_2(t)+\gamma\hbar\theta_2(t)]\psi_2(t)+K\psi_1(t) \tag{3.14}$$

where $\theta_\sharp(t)=\arg\psi_\sharp(t)(\sharp=1,2)$ and we have introduced heuristically an amplitude K to penetrate the layer of insulator.

Under the substitutions

$$\begin{aligned} \psi_1(t)&=[\rho_1(t)]^{1/2}\exp[i\theta_1(t)] \\ \psi_2(t)&=[\rho_2(t)]^{1/2}\exp[i\theta_2(t)] \end{aligned} \tag{3.15}$$

where $\rho_\sharp(t)=|\psi_\sharp(t)|^2$ denotes a total number of Cooper pairs in the region

$\sharp = 1, 2$, equations (3.13) and (3.14) yield

$$\dot{\rho}_1(t) = \frac{2}{\hbar} K[\rho_1(t)\rho_2(t)]^{1/2} \sin[\theta_2(t) - \theta_1(t)] \tag{3.16}$$

$$\dot{\rho}_2(t) = -\frac{2}{\hbar} K[\rho_1(t)\rho_2(t)]^{1/2} \sin[\theta_2(t) - \theta_1(t)] \tag{3.17}$$

$$\dot{\theta}_1(t) = \frac{K}{\hbar}\left[\frac{\rho_2(t)}{\rho_1(t)}\right]^{1/2} \cos[\theta_2(t)\theta_1(t)] - \gamma\theta_1(t) - \frac{q}{\hbar} V_1(t) - \frac{1}{\hbar} R_1(t) \tag{3.18}$$

$$\dot{\theta}_2(t) = \frac{K}{\hbar}\left[\frac{\rho_1(t)}{\rho_2(t)}\right]^{1/2} \cos[\theta_2(t) - \theta_1(t)] - \gamma\theta_2(t) - \frac{q}{\hbar} V_2(t) - \frac{1}{\hbar} R_2(t) \tag{3.19}$$

Let ρ_0 be a constant total number of Cooper pairs in each superconducting region and put

$$\begin{aligned} \rho_1(t) &= \rho_0 + \rho(t) \\ \rho_2(t) &= \rho_0 - \rho(t) \end{aligned} \tag{3.20}$$

with $\rho(t) \ll \rho_0$, then equations (3.16)–(3.19) yield the following equations up to the first order in $\rho(t)$:

$$\dot{\rho}(t) = \frac{2}{\hbar} K\rho_0 \sin[\theta_2(t) - \theta_1(t)] \tag{3.21}$$

$$\dot{\theta}_1(t) = \frac{K}{\hbar} \cos[\theta_2(t) - \theta_1(t)] - \gamma\theta_1(t) - \frac{q}{\hbar} V_1(t) - \frac{1}{\hbar} R_1(t) \tag{3.22}$$

$$\dot{\theta}_2(t) = \frac{K}{\hbar} \cos[\theta_2(t) - \theta_1(t)] - \gamma\theta_2(t) - \frac{q}{\hbar} V_2(t) - \frac{1}{\hbar} R_2(t) \tag{3.23}$$

By those equations we find the total current across the junction to be

$$\begin{aligned} J(t) &= \dot{\rho}(t) \\ &= \frac{2}{\hbar} K\rho_0 \sin\theta(t) \\ &= J_0 \sin\theta(t) \end{aligned} \tag{3.24}$$

where $\theta(t)=\theta_2(t)-\theta_1(t)$ satisfies a stochastic differential equation

$$\dot{\theta}(t)=-\gamma\theta(t)+\frac{q}{\hbar}V(t)+\frac{1}{\hbar}R(t) \tag{3.25}$$

with initial condition $\theta(0)=\theta_0$.

The solution of equation (3.25) is known to be a Gaussian random process $\theta=\Theta(t)$ with mean

$$m(t)=\theta_0 e^{-\gamma t}+\frac{q}{\hbar}e^{-\gamma t}\int_0^t V(u)e^{\gamma u}\,du \tag{3.26}$$

and variance

$$\sigma(t)=\frac{G}{\gamma\hbar^2}(1-e^{-2\gamma t}) \tag{3.27}$$

Correspondingly the total current across the junction becomes a random process

$$J(t)=J_0\sin\Theta(t) \tag{3.28}$$

The observed current $\langle J(t)\rangle_{\text{ob}}$ should be an average of equation (3.28), which can be calculated as

$$\begin{aligned}\langle J(t)\rangle_{\text{ob}}&=\mathbb{E}\{J(t)\}\\&=J_0\mathbb{E}\{\sin\Theta(t)\}\\&=J_0\,\text{Im}\,\mathbb{E}\{\exp i\Theta(t)\}\\&=J_0\sin[m(t)]\exp[-\sigma(t)]\end{aligned} \tag{3.29}$$

where we have used the characteristic function of $\Theta(t)$

$$\mathbb{E}\{\exp i\Theta(t)x\}=\exp[-\sigma(t)x^2+\text{Im}(t)x] \tag{3.30}$$

for $x\in\mathbb{R}$. Namely, we have the following expression for the observed total current across the junction:

$$\langle J(t)\rangle_{\text{ob}}=J_0\sin\left[\theta_0 e^{-\gamma t}+\frac{g}{\hbar}e^{-\gamma t}\int_0^t V(u)e^{\gamma u}\,du\right]\exp\left[-\frac{G}{\gamma\hbar^2}(1-e^{-2\gamma t})\right] \tag{3.31}$$

If we put on an AC voltage

$$V(t)=V_0+v\cos\omega t \tag{3.32}$$

with $v\ll V_0$ between two superconductors, the observed current across the junction becomes

$$\begin{aligned}\langle J(t)\rangle_{\text{ob}}&=J_0\sin\left[\theta_0 e^{-\gamma t}+\frac{qV_0}{\gamma\hbar}(1-e^{-\gamma t})\right.\\&\left.+\frac{\gamma}{\gamma^2+\omega^2}\cdot\frac{qv}{\hbar}\left(\cos\omega t+\frac{\omega}{\gamma}\sin\omega t-e^{-\gamma t}\right)\right]\exp\left[-\frac{G}{\gamma\hbar^2}(1-e^{-2\gamma t})\right]\\&\simeq J_0\left\{\sin\left[\theta_0 e^{-\gamma t}+\frac{qV_0}{\gamma\hbar}(1-e^{-\gamma t})\right]\frac{\gamma}{\gamma^2+\omega^2}\frac{qV}{\hbar}\left(\cos\omega t+\frac{\omega}{\gamma}\sin\omega t-e^{-\gamma t}\right)\right.\\&\left.\times\cos\left[\theta_0 e^{-\gamma t}+\frac{qV_0}{\gamma\hbar}(1-e^{-\gamma t})\right]\right\}\exp\left[-\frac{G}{\gamma\hbar^2}(1-e^{-2\gamma t})\right]\end{aligned} \tag{3.33}$$

This expression for the observed superconducting current at a finite temperature coincides with the one at absolute zero temperature (Feynman et al., 1975; Feynman, 1972) in the limit $\gamma\to 0, G\to 0$. For sufficiently large $t(t\gg 1/\gamma)$ we have

$$\langle J(t)\rangle_{\text{ob}}\sim J_0\left[\sin\frac{qV_0}{\gamma\hbar}+\frac{\gamma}{\gamma^2+\omega^2}\frac{qv}{\hbar}\left(\cos\omega t+\frac{\omega}{\gamma}\sin\omega t\right)\cos\frac{qV_0}{\gamma\hbar}\right]\exp\left(-\frac{G}{\gamma\hbar^2}\right) \tag{3.34}$$

which has nonvanishing Cesàro mean

$$\langle J(t)\rangle_{\text{ob}}\underset{t\gg 1/\gamma}{\sim} J_0\sin\left(\frac{qV_0}{\gamma\hbar}\right)\exp\left(-\frac{G}{\gamma\hbar^2}\right) \tag{3.35}$$

Thus we find that the observed AC Josephson current at a finite temperature $T\simeq T_c$ approaches a stationary equilibrium value independent of the initial condition and the AC frequency.

In our approach three free parameters J_0, γ, and G are left unspecified. To specify them in each practical case as Ford, Kac, and Mazur (1965) is still an open problem.

3.3. Viscous Quantum Fluid of Nucleonic Matter. As a practical problem, there are two typical examples of dissipative fields; viscous quantum fluids of nucleonic matters and the laser electric field in the lossy cavity. The former will be studied in this section and the latter in the next one by making use of the Schrödinger–Langevin equation.

It is known that there exist certain dissipative phenomena in the nuclear collective dynamics in which frictions or viscosities of nucleonic matters take part (Griffin and Kan, 1976).

Suppose we are mainly concerned with the long-range behavior of the nucleon collective motions, and so may be allowed to adopt the hydrodynamical treatment. Namely, the proton fluid with density ρ_p and neutron fluid with ρ_n are assumed to interpenetrate under the influence of the nuclear binding energy $K(N-Z)^2/A$, where A is the mass number, N the neutron number, Z the proton number, and $K \simeq 20$ MeV. The short-range property of the nucleon–nucleon interactions yields the nuclear binding energy per nucleon as given by $K(\rho_n - \rho_p)^2/\rho_0$ with constant total density $\rho_0 = \rho_n + \rho_p$.

Under those assumptions the proton–neutron density difference $\rho = \rho_p - \rho_n$ and relative velocity $\mathbf{v} = \mathbf{v}_p - \mathbf{v}_n$ are considered as those of a viscous fluid. It is described by the Navier–Stokes equation

$$\rho\dot{\mathbf{v}} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -u^2\nabla\rho + \eta\Delta\mathbf{v} \tag{3.36}$$

and the equation of continuity

$$\dot{\rho} + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{3.37}$$

where $u = (8KZN/MA^2)^{1/2}$ is the sound velocity (M denotes the nucleon mass) and η the viscosity. As we are interested in the quantum fluctuation of the relative density disturbance, it is enough to consider a linearized equation

$$\ddot{\rho} - u^2\Delta\rho - \nu\Delta\dot{\rho} = 0 \tag{3.38}$$

where ν denotes the kinematic viscosity.

Since $\rho(\cdot,t) \in L_2(\mathbb{R}^3, d^3r) = L_2(S^2, d\Omega) \otimes L_2([0,\infty], r^2\,dr)$, it is convenient to expand $\rho(\cdot,t)$ in terms of C.N.O.S. $\{Y_{lm} \otimes j_l\}_{l=0}^{\infty}{}_{m=-l}^{l}$, i.e.,

$$\rho(\mathbf{r},t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm}(t)\, Y_{lm}(\theta,\varphi)\, j_l\!\left(\frac{r}{\Lambda}\right), \tag{3.39}$$

where the Y_{lm}'s are spherical harmonics and j_l's spherical Bessel functions, respectively.

We shall take only the nuclear dipole vibration ($l=1, m=0$) into account, obtaining

$$\ddot{a}(t) = -\frac{u^2}{\Lambda^2} a(t) - \frac{\nu}{\Lambda^2} \dot{a}(t) \tag{3.40}$$

where we have made an abbreviation $a(t)$ for $A_{10}(t)$. The constant Λ should be determined by a boundary condition

$$j_1(\text{nuclear radius}/\Lambda) = 0 \tag{3.41}$$

As we showed in Section 9, a quantum mechanical behavior of the nuclear viscous dipole vibration (3.40) would be well described by the Schrödinger–Langevin equation

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \psi(a,t) &= -\frac{\hbar^2}{2} \frac{\partial^2}{\partial a^2} \psi(a,t) + \frac{1}{2}\left(\frac{u}{\Lambda}\right)^2 a^2 \psi(a,t) \\ &\quad + \frac{i\hbar}{2} \frac{\nu}{\Lambda^2} \log\left[\frac{\bar{\psi}(a,t)}{\psi(a,t)}\right] \psi(a,t) \\ &= \left[-\frac{\hbar^2}{2} \frac{\partial^2}{\partial a^2} + \frac{1}{2}\left(\frac{u}{\Lambda}\right)^2 a^2 + \frac{i\hbar}{2} \frac{\nu}{\Lambda^2} \log \frac{\bar{\psi}(a,t)}{\psi(a,t)}\right] \psi(a,t) \end{aligned} \tag{3.42}$$

It can be seen easily that the minimum uncertainty state around a classical viscous dipole vibration $a_c(t) = a \exp i\omega t$, $\omega^2 - (u/\Lambda)^2 - i\omega\nu/\Lambda^2 = 0$,

$$\psi_c(a,t) = h_0[a - a_c(t)] \exp\left\{-\frac{i}{\hbar}\left[\frac{\hbar u}{2\Lambda} t + \dot{a}_c(t) + \int_0^t \frac{1}{2} \dot{a}_c(s)^2 \, ds\right]\right\} \tag{3.43}$$

is a special solution of the Schrödinger—Langevin equation (3.42), where $h_0 \in L_2(\mathbb{R})$ denotes a harmonic oscillator ground state wave function (Yasue, 1976; Kan and Griffin, 1974; Skagerstam, 1975). Asymptotic behavior of the characteristic state (3.43),

$$\lim_{t \to \infty} \|\psi(\cdot, t) - h_0(\cdot)\| = 0 \tag{3.44}$$

also indicates a dissipative property of the Schrödinger–Langevin equation (3.42).

3.4 Lossy Laser. In the case of lossy laser, Maxwell's equations in MKS units

$$\begin{aligned} &\nabla \cdot \mathbb{D}=0, &&\mathbb{D}=\varepsilon_0\mathbb{E}+\mathbb{P} \\ &\nabla \cdot \mathbb{B}=0, &&\mathbb{B}=\mu_0\mathbb{H} \\ &\nabla \times \mathbb{E}=-\dot{\mathbb{B}}, &&\nabla \times \mathbb{H}=\mathbf{j}+\dot{\mathbb{D}} \end{aligned} \tag{3.45}$$

and a phenomenological Ohmic loss relation

$$\mathbf{j}=\sigma\mathbb{E} \tag{3.46}$$

give us the following dissipative field equation:

$$-\Delta\mathbb{E}+\mu_0\sigma\dot{\mathbb{E}}+\mu_0\varepsilon_0\ddot{\mathbb{E}}=-\mu_0\ddot{\mathbb{P}} \tag{3.47}$$

The laser electric field $\mathbb{E}$ is forced by the imposed polarization vector $\mathbb{P}$ and damped by the Ohmic energy loss in the cavity through equation (3.47). Equation (3.47) completely determines the laser electric field in the lossy cavity without detailed description of the mechanism of the cavity loss. The conventional treatment of such a lossy laser was to describe the atoms in a laser quantum mechanically and the electric field (3.47) classically (Rogovin and Scully, 1976). Therefore quantum mechanical description of the laser electric field in the lossy cavity seems to be needed.[1]

In this section, we shall investigate quantum mechanics of the laser electric field described by the dissipative field equation (3.47) in much detail.

For simplicity, let us assume the laser electric field is linearly polarized, i.e.,

$$\mathbb{E}(\mathbf{r},t)=\mathbf{e}E(\mathbf{r},t) \tag{3.48}$$

for some unit vector $\mathbf{e}$. Then equation (3.47) becomes

$$\ddot{E}=c^2\Delta E-\frac{\sigma}{\varepsilon_0}\dot{E}-\frac{1}{\varepsilon_0}\mathbf{e}\cdot\ddot{\mathbb{P}} \tag{3.49}$$

where c stands for the light velocity.

Quantum mechanical behavior of the laser electric field (3.49) is characterized by the Schrödinger–Langevin equation, as we have verified

[1] A constructive approach to quantum mechanics of the lossy laser from a fundamental point of view was given by Hepp and Lieb (1973). They made use of the Heisenberg representation, whereas our standpoint, explained here, may be understood as the Schrödinger representation.

in Sections 8 and 9,

$$i\hbar\frac{\partial}{\partial t}\Omega(E,t)=\frac{1}{2}\int_{\mathcal{C}}d^3r\left[-\hbar^2\frac{\delta^2}{\delta E(\mathbf{r})^2}+c^2|\nabla E(\mathbf{r})|^2\right]\Omega(E,t)$$
$$+\frac{i\hbar\sigma}{2\varepsilon_0}\log\frac{\overline{\Omega}(E,t)}{\Omega(E,t)}\cdot\Omega(E,t)+\frac{1}{\varepsilon_0}\int_{\mathcal{C}}d^3rE(\mathbf{r})\mathbf{e}\cdot\ddot{\mathbb{P}}(\mathbf{r},t)\Omega(E,t) \tag{3.50}$$

To further simplify the analysis, we take a C.N.O.S. $\{e_n\}_{n\in\mathbb{N}}\subset\mathcal{S}(\mathcal{C})$, where $\mathcal{C}\subset\mathbb{R}^3$ denotes the cavity region, to be eigenfunctions of the Laplacian Δ. Namely, we have

$$\Delta e_n(\mathbf{r})=-k_n^2e_n(\mathbf{r}),\qquad n\in\mathbb{N},\qquad \mathbf{r}\in\mathcal{C} \tag{3.51}$$

where $-k_n^2$'s are eigenvalues.

As the quantized electric field $E(\mathbf{r})\in{}^*\mathcal{S}(\mathbb{R}^3)$ and the state functional $\Omega(E,t)$ are decomposed as $E(\mathbf{r})={}^*[\Sigma_{p\leqslant n}a_p^{(n)}e_p(\mathbf{r})]$ and $\Omega(E,t)={}^*[\Omega_n(a^{(n)},t)]$, respectively, we can rewrite equation (3.50) as

$$^*\left[i\hbar\frac{\partial}{\partial t}\Omega_n(a^{(n)},t)\right]={}^*\left[-\frac{\hbar^2}{2}\sum_{p<n}\left\{\frac{2}{\partial a_p^{(n)2}}+\frac{c^2k_p^2}{2}a_p^{(n)2}\right\}\Omega_n(a^{(n)},t)\right]$$
$$+{}^*\left[\frac{i\hbar\sigma}{2\varepsilon_0}\log\frac{\overline{\Omega}_n(a^{(n)},t)}{\Omega_n(a^{(n)},t)}\cdot\Omega_n(a^{(n)},t)\right]$$
$$+\frac{1}{\varepsilon_0}{}^*\left[\sum_{p<n}a_p^{(n)}\ddot{P}_p(t)\Omega_n(a^{(n)},t)\right] \tag{3.52}$$

where we have made the the abbreviation $P_p(t)=\int e_p(\mathbf{r})\mathbf{e}\cdot\mathbb{P}(\mathbf{r},t)d^3r$. To solve the functional differential equation (12.8), it is enough to consider its finite-dimensional cross section ("cross section" denotes the finite-dimensional element inside the asterisk bracket *[ ]). If one puts

$$\Omega_n(a^{(n)},t)=\prod_{p<n}\Omega(a_p^{(n)},t) \tag{3.53}$$

equation (3.52) reduces to the following one-dimensional Schrödinger–

Langevin equations:

$$i\hbar\frac{\partial}{\partial t}\Omega\left(a_p^{(n)},t\right)=\left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial a_p^{(n)2}}+\frac{c^2k_p^2}{2}a_p^{(n)2}\right)\Omega\left(a_p^{(n)},t\right)$$
$$+\frac{i\hbar\sigma}{2\varepsilon_0}\log\frac{\bar{\Omega}\left(a_p^{(n)},t\right)}{\Omega\left(a_p^{(n)},t\right)}\cdot\Omega\left(a_p^{(n)},t\right)+\frac{1}{\varepsilon_0}a_p^{(n)}\ddot{P}_p(t)\Omega\left(a_p^{(n)},t\right)$$
$$(1\leqslant p\leqslant n) \quad (3.54)$$

A characteristic solution to equation (3.54) is obtained by introducing the so-called photon coherent state $\Omega_c(a_p^{(n)};z)$, $z\in\mathbb{C}$ ($\mathbb{C}$ denotes the complex plane). The coherent state is defined as an eigenstate of the annihilation operator (Klauder and Sudarshan, 1968):

$$\left(\frac{1}{2\hbar k_p c}\right)^{1/2}\left[\frac{\partial}{\partial a_p^{(n)}}+ck_pa_p^{(n)}\right]\Omega_c\left(a_p^{(n)};z\right)=z\Omega_c\left(a_p^{(n)};z\right) \quad (3.55)$$

Namely, the coherent state

$$\Omega_c\left(a_p^{(n)};\alpha_p(t)-\frac{i}{ck_p}\dot{\alpha}_p(t)\right)\exp\left\{-\frac{1}{2\hbar ck_p}\dot{\alpha}_p(t)^2-\frac{i}{\hbar}\left[\alpha_p(t)\dot{\alpha}_p(t)+g_p(t)\right]\right\}$$
$$(3.56)$$

solves equation (3.54), provided that $\alpha_p(t)$ satisfies the classical equation of motion

$$\ddot{\alpha}_p(t)=-\frac{\sigma}{\varepsilon_0}\dot{\alpha}_p(t)-c^2k_p^2\alpha_p(t)-\frac{1}{\varepsilon_0}\ddot{P}_p(t) \quad (3.57)$$

and $g_p(t)$ is related to $\alpha_p(t)$ as

$$\dot{g}_p(t)+g_p(t)=\frac{\hbar ck_p}{2}+\frac{\hbar c^2k_p^2}{2}\alpha_p(t)^2-\frac{1}{2}\dot{\alpha}_p(t)^2 \quad (3.58)$$

Correspondingly, a quantum mechanical behavior of the laser electric field in the lossy cavity can be represented by the photon coherent state

$$\Omega_c(E,t)={}^*\left[\Omega_c(a^{(n)},t)\right]$$
$$={}^*\left[\prod_{p<n}\Omega_c\left(a_p^{(n)};\alpha_p(t)-\frac{i}{ck_p}\dot{\alpha}_p(t)\right)\right.$$
$$\left.\exp\left\{-\frac{1}{2\hbar ck_p}\dot{\alpha}_p(t)^2-\frac{i}{\hbar}\left[\alpha_p(t)\dot{\alpha}_p(t)+g_p(t)\right]\right\}\right] \quad (3.59)$$

Therefore we may be allowed to mention that the quantized laser electric field in the lossy cavity fluctuates around its classical value $E_c(\mathbf{r},t)=$ *$[\Sigma_{p\leqslant n}\alpha_p(t)e_p(\mathbf{r})]$ with minimum uncertainty. This provides a quantum theoretical background to the validity of the conventional semiclassical treatment of the lossy laser.

4. TUNNEL EFFECT IN NON-ABELIAN GAUGE THEORY

Originated from Polyakov's (1975) work, several authors (Callen et al., 1976, 1977; 't Hooft, 1976; Jackiw and Rebbi, 1976) have investigated a Euclidean path integral description of vacuum tunneling phenomena in non-Abelian gauge field theory. They suggested that a Euclidean path integral

$$\int \exp\left(\frac{-1}{\hbar} S_E[A] + \text{gauge-fixing term}\right)\delta A$$

$$S_E[A] = \frac{1}{4}\sum_{a=1}^{3}\sum_{\mu,\nu=0}^{3}\int F_{\mu\nu}^{a}F_{\mu\nu}^{a}\,d^4x$$

provides a powerful tool to explore the structure of gauge theory vacuum. Classical Euclidean solutions, which minimize the Euclidean action $S_E[A]$, were shown to manifest tunneling phenomena between topologically inequivalent classical vacua within the realm of the WKB approximation.

Although application of such a Euclidean technique to the problem of vacuum instability has been put into practice successfully (Coleman, 1977; Callan and Coleman, 1977; Pak, 1977; Creutz and Tudron, 1977; Banks et al., 1973; Bitar and Chang, 1978; Gildener and Patrascioiu, 1977; Jackiw, 1977), there have been no rigorous arguments which explain why the Euclidean path integral is relevant for describing the vacuum tunneling phenomena in the physical space–time (i.e., Minkowski space), except a heuristic one based on the WKB method ('t Hooft, 1976).

In this chapter, the problem of vacuum tunneling phenomena such as the quantum decay process of metastable vacuum states in $SU(2)$ Yang–Mills theory will be investigated from a probability theoretical point of view. The mechanism of the vacuum tunneling can be illustrated within the realm of the stochastic quantization.

The main sources for this chapter were the present author's papers (Yasue, 1978c, d).

4.1. Euclidean Path Integral Description of the Vacuum Tunneling Phenomena. In classical field theory, the $SU(2)$ Yang–Mills field A_μ is nothing but a $SU(2)$ Lie module over the space–time. Since $A_\mu(\mathbf{x},t)$ belongs to the Lie algebra of $SU(2)$ for each space–time point $(\mathbf{x},t)$, it has an isovector expression

$$A_\mu(\mathbf{x},t)=\sum_{a=1}^{3} A_\mu^a(\mathbf{x},t)T^a \tag{4.1}$$

where $\{T^a\}_{a=1}^3$ is a basis of the Lie algebra of $SU(2)$.

Dynamics of the Yang–Mills field is given by a Lagrangean

$$L=-\frac{1}{4}\sum_{a=1}^{3}\sum_{\mu,\nu=0}^{3}{}' \int F_{\mu\nu}^a F_{\mu\nu}^a\, d^3x \tag{4.2}$$

where

$$F_{\mu\nu}^a=\partial_\mu A_\nu^a-\partial_\nu A_\mu^a+\sum_{b,c=1}^{3}\varepsilon^{abc}A_\mu^b A_\nu^c \tag{4.3}$$

is a field strength tensor. (Latin letters a,b,c denote isovector indices and i,j,k denote space indices. Greek letters μ,ν denote space–time indices, and the terms in the primed sum for $\mu,\nu=1,2,3$ are taken with reversed sign.) In terms of "electromagnetic" fields

$$E_i^a=F_{0i}^a,\qquad B_i^a=\frac{1}{2}\sum_{k,j=1}^{3}\varepsilon_{ijk}F_{jk}^a \tag{4.4}$$

the Lagrangean (4.2) can be written as

$$L=\frac{1}{2}\sum_{i,a=1}^{3}\int (E_i^aE_i^a-B_i^aB_i^a)\,d^3x \tag{4.5}$$

Hereafter, to avoid the complexity of the Coulomb gauge (Gribov, 1977; Hirayama et al., 1978) we shall work in $A_0=0$ gauge. Then E_i^a becomes identical with $\partial_0 A_i^a=\dot{A}_i^a$, and B_i^a does not contain $\dot{A}_i^a$. In this gauge A_i^a's are dynamical variables of the Yang–Mills field. The Lagrangean (4.5) becomes

$$L=\frac{1}{2}\sum_{i,a=1}^{3}\int \left(\dot{A}_i^a\dot{A}_i^a-B_i^aB_i^a\right)d^3x \tag{4.6}$$

which gives us the following equation of motion for A_i^a:

$$\ddot{A}_i^a = -\frac{\delta}{\delta A_i^a}\frac{1}{2}\int B_j^b B_j^b\, d^3y \tag{4.7}$$

(From now on, the summation convention for all repeated Latin indices is assumed.) To quantize the Yang–Mills field, it is convenient to adopt the stochastic quantization procedure. This is because not only the structure of the vacuum state but also the mechanism of the vacuum tunneling of the quantized Yang–Mills field can be illustrated within the realm of the stochastic quantization.

All the quantization procedures are the same as in Section 8, provided that the field equation in the generalized sense (2.31) is replaced by the following one:

$$\frac{1}{2}(DD_* + D_*D)A_i^a(t) = -\frac{\delta}{\delta A_i^a(t)}\frac{1}{2}\int B_j^b B_j^b\, d^3y \tag{4.8}$$

The quantized Yang–Mills field $A_i^a(t), -\infty < t < \infty$, is, of course, a random process in $^*\mathscr{E}(\mathbb{R}^3)$ of the aforementioned type. The resulting Schrödinger equation is

$$i\hbar\frac{\partial}{\partial t}\Omega(\mathbf{A},t) = \frac{1}{2}\int d^3x\left(-\hbar^2\frac{\delta^2}{\delta A_i^a \delta A_i^a} + B_i^a B_i^a\right)\Omega(\mathbf{A},t) \tag{4.9}$$

where $\mathbf{A}$ is an abbreviation for $\{A_i^a\}_{i,a=1}^3$.

We shall investigate a structure of the vacuum state by making use of a probability theoretical framework of the stochastic quantization in what follows.

Removing the infinite zero-point energy, we define a quantum theoretical vacuum state of the Yang–Mills theory by a ground state wave functional $\Omega(\mathbf{A})$ on $^*\mathscr{E}(\mathbb{R}^3)$ which satisfies the Schrödinger equation

$$\int d^3x\left(-\frac{\hbar^2}{2}\frac{\delta^2}{\delta A_i^a \delta A_i^a} + \frac{1}{2}B_i^a B_i^a\right)\Omega(\mathbf{A}) = 0 \tag{4.10}$$

In classical field theory, vacuum states of the Yang–Mills theory are classical field configurations $'A_i^a(\mathbf{x})$ with zero potential energy

$$\frac{1}{2}\int B_i^a('\mathbf{A})B_i^a('\mathbf{A})\, d^3x = 0 \tag{4.11}$$

They are pure gauge fields

$$'A_i = 'g^{-1}\partial_i 'g \in SU(2) \text{ Lie module on } \mathbb{R}^3 \tag{4.12}$$

where $'g$'s are unitary matrices such that $\lim_{|\mathbf{x}|\to\infty} 'g(\mathbf{x}) = I$. As $\lim_{|\mathbf{x}|\to\infty} 'A(\mathbf{x}) = 0$, we can consider the pure gauge fields $'\mathbf{A}$ continuous mappings from $\dot{\mathbb{R}}^3$ to $SU(2)$. Since $\dot{\mathbb{R}}^3 \cong S^3$ (three-dimensional sphere) and also $SU(2) \cong S^3$, $'A$'s can be classified by the third Homotopy group of S^3:

$$\pi_3(S^3) \cong \mathbb{Z} \qquad \text{(integers)} \tag{4.13}$$

with respect to a fixed point $\infty \in \dot{\mathbb{R}}^3$. Namely, classical vacuum states of the Yang–Mills theory consist of an infinite number of homotopy classes of S^3:

$$[\,'A_i = 'g^{-1}\partial_i 'g\,], [\,''A_i = ''g^{-1}\partial_i ''g\,], \ldots, \tag{4.14}$$

where $[\cdot]$ denotes a homotopy class to which the pure gauge field inside the bracket belongs. Two pure gauge fields which can be joined with each other by a continuous gauge transformation in the manifold of $SU(2)$ Lie module should be understood as the same classical vacuum state (Jackiw, 1977).

In order to classify the homotopy classes, it is convenient to introduce the Pontryagin index

$$q = -\frac{1}{24\pi^2} \sum_{i,j,k=1}^{3} \int \operatorname{Tr}(A_i A_j A_k)\, d^3x \tag{4.15}$$

For pure gauge fields, q is an integer which belongs to the homotopy group $\pi_3(S^3) \cong \mathbb{Z}$. Then the classical vacuum states (4.14) can be rearranged as

$$\{[\,{}^qA_i = {}^qg^{-1}\partial_i {}^qg\,]\}_{q\in\mathbb{Z}} \tag{4.16}$$

where ${}^qA_i$ is a pure gauge field with Pontryagin index q.

In quantum field theory the classical vacuum states (4.16) are rendered unstable by the tunnel effect; they are metastable vacuum states.

Let us investigate the vacuum tunneling phenomena between metastable vacuum states ${}^qA_i$'s from a probability theoretical point of view.

In the conventional framework of quantum field theory, the wave functional $\Omega(\mathbf{A})$ does not teach us the details of the vacuum tunneling. One can describe the tunneling behavior of the quantized Yang–Mills field only in the semiclassical limit, i.e., within the realm of the WKB approximation.

In the probability theoretical framework of the stochastic quantization, on the contrary, the behavior of the quantized Yang–Mills field in the vacuum state $\Omega(A)$ is known to be a random process $A_i^a(t)$ in $*\mathfrak{S}(\mathbb{R}^3)$ as we have seen in Section 8. By equations (1.41), (2.33), and (2.34) we find that the random process $A_i^a(t), -\infty<t<\infty$, is a solution of the stochastic differential equation

$$dA_i^a(t) = U_i^a(\mathbf{A}(t))\,dt + dW_i^a(t) \tag{4.17}$$

where the transformation $U_i^a(\cdot)$ on $*\mathfrak{S}(\mathbb{R}^3)$ is related with the vacuum state wave functional by

$$U_i^a(\mathbf{A}) = \hbar\frac{\delta}{\delta A_i^a}\log\Omega(\mathbf{A}) \tag{4.18}$$

and $W_i^a(t)$ denotes a Wiener process in $*\mathfrak{S}(\mathbb{R}^3)$ with diffusion constant $\hbar/2$. Notice that the stochastic differential equation (4.17) is an abbreviation for the relation

$$A_i^a(t) - A_i^a(s) = \int_s^t U_i^a(\mathbf{A}(u))\,du + W_i^a(t) - W_i^a(s), \qquad t>s \tag{4.19}$$

Therefore, a transition probability law of the random process $A_i^a(t), -\infty<t<\infty$, manifests the tunneling process of the quantized vacuum field configuration between metastable vacuum states ${}^qA_i$. The transition probability law is given by an elementary solution $Q[\mathbf{A};s|'\mathbf{A};u]$ of the Fokker–Planck equation (2.34). Namely, we have

$$\frac{\partial}{\partial s}Q = -\int d^3x\frac{\delta}{\delta A_i^a}\left[U_i^a(\mathbf{A})Q\right] + \frac{\hbar}{2}\int d^3x\frac{\delta^2}{\delta A_i^a\delta A_i^a}Q \tag{4.20}$$

and

$$\lim_{s\downarrow u} Q\left[\mathbf{A};s|'\mathbf{A};u\right] = \delta(\mathbf{A}-'\mathbf{A}) \tag{4.21}$$

where $\delta(\cdot)$ denotes a delta functional on $*\mathfrak{S}(\mathbb{R}^3)$.

To illustrate the mechanism of the vacuum tunneling, one needs to solve the Cauchy problem equations (4.20) and (4.21). This can be done by introducing a relative transition law $F[\mathbf{A};s|'\mathbf{A};u]$ by

$$Q\left[\mathbf{A};s|'\mathbf{A};u\right] = \Omega(\mathbf{A})F\left[\mathbf{A};s|'\mathbf{A};u\right]\Omega('A)^{-1} \tag{4.22}$$

Equation (4.20) is transformed into a self-adjoint form

$$-\hbar\frac{\partial}{\partial s}F=\int d^3x\left(-\frac{\hbar^2}{2}\frac{\delta^2}{\delta A_i^a\delta A_i^a}+\frac{1}{2}B_i^aB_i^a\right)F \tag{4.23}$$

and the initial condition (4.21) into

$$\lim_{s\downarrow u}F[\mathbf{A};s|'\mathbf{A};u]=\delta(\mathbf{A}-'\mathbf{A}) \tag{4.24}$$

by the substitution (4.22). Equation (4.23) is nothing but a Euclidean analog of the Schrödinger equation (4.9). The Feynman–Kac formula (1.18) and (1.19) asserts that a solution to the Cauchy problem (4.23) and (4.24) is given by a Wiener integral

$$F[\mathbf{A};s|'\mathbf{A};u]=\int\exp\left\{-\frac{1}{2\hbar}\int_u^s dt\int d^3xB_i^a[\mathbf{A}(t)]B_i^a[\mathbf{A}(t)]\right\}$$
$$\times\,\delta[\mathbf{A}(s)-\mathbf{A}]^*\mu_w^{u,'\mathbf{A}}[\delta\mathbf{A}(\cdot)] \tag{4.25}$$

Here $^*\mu_w^{u,'\mathbf{A}}$ denotes a nonstandard Wiener measure with diffusion constant $\hbar/2$ defined on the totality of continuous paths $\mathbf{A}(\cdot)$ in $^*\mathfrak{S}(\mathbb{R}^3)$ starting from $'\mathbf{A}$ at $t=u$. Correspondingly the transition probability law of the random process $A_i^a(t)$, $-\infty<t<\infty$, is found to be

$$Q[\mathbf{A};s|'\mathbf{A};u]=\frac{\Omega(\mathbf{A})}{\Omega('A)}\int\exp\left\{-\frac{1}{2\hbar}\int_u^s dt\int d^3xB_i^a[\mathbf{A}(t)]B_i^a[\mathbf{A}(t)]\right\}$$
$$\times\,\delta[\mathbf{A}(s)-\mathbf{A}]^*\mu_w^{u,'\mathbf{A}}[\delta\mathbf{A}(\cdot)] \tag{4.26}$$

A tunneling probability of the quantized vacuum field configuration between metastable vacuum states ^qA_i and ^pA_i, $q\neq p$, from a remote past to a remote future is $Q[^p\mathbf{A};\infty|^q\mathbf{A};-\infty]$. Noticing that $Q[^p\mathbf{A};\infty|^q\mathbf{A};-\infty]$ would coincide with the conventional expression of the tunneling probability, i.e., the ratio $|\Omega(^p\mathbf{A})/\Omega(^q\mathbf{A})|^2$, we find a vacuum tunneling amplitude of the quantized Yang–Mills field in $A_0=0$ gauge to be

$$\mathrm{Amp}[^p\mathbf{A};\infty|^q\mathbf{A};-\infty]=\Omega(^p\mathbf{A})/\Omega(^q\mathbf{A})$$
$$=\int\exp\left\{-\frac{1}{2\hbar}\int_{-\infty}^{\infty}dt\int d^3xB_i^a[\mathbf{A}(t)]B_i^a[\mathbf{A}(t)]\right\}$$
$$\times\,\delta[\mathbf{A}(\infty)-{}^p\mathbf{A}]^*\mu_w^{-\infty,^q\mathbf{A}}[\delta\mathbf{A}(\cdot)] \tag{4.27}$$

If we introduce a functional path integral expression of the Wiener integral

$$\int \delta[\mathbf{A}(\infty)-{}^{p}\mathbf{A}]\,{}^{*}\mu_{w}^{-\infty,{}^{q}\mathbf{A}}[\delta\mathbf{A}(\cdot)]\cdots$$
$$= N\cdot \int_{{}^{q}\mathbf{A}}^{{}^{p}\mathbf{A}} \exp\left[-\frac{1}{2\hbar}\int_{-\infty}^{\infty} dt \int d^3x \dot{A}_i^a(t)\dot{A}_i^a(t)\right]\delta\mathbf{A}(\cdot)\cdots \tag{4.28}$$

where $\delta\mathbf{A}(\cdot)$ means to take a functional path integral and $N \in {}^{*}\mathbb{R}$ is a normalization constant, equation (4.27) becomes

$$\mathrm{Amp}[{}^{p}\mathbf{A};\infty|{}^{q}\mathbf{A};-\infty] = N\cdot \int_{{}^{q}\mathbf{A}}^{{}^{p}\mathbf{A}} \exp\left\{-\frac{1}{\hbar}\int_{-\infty}^{\infty} dt \int d^3x \left[\frac{1}{2}\dot{A}_i^a(t)\dot{A}_i^a(t) + \frac{1}{2}B_i^a(A(t))B_i^a(A(t))\right]\right\}\delta\mathbf{A}(\cdot) \tag{4.29}$$

In terms of the field strength tensor $F^a_{\mu\nu}$, this can be written as

$$\mathrm{Amp}[{}^{p}\mathbf{A};\infty|{}^{q}\mathbf{A};-\infty] = N\cdot \int_{{}^{q}\mathbf{A}}^{{}^{p}\mathbf{A}} \exp\left(-\frac{1}{4\hbar}\sum_{a=1}^{3}\sum_{\mu,\nu=0}^{3}\int F^a_{\mu\nu}F^a_{\mu\nu}d^4x\right)\delta\mathbf{A}(\cdot) \tag{4.30}$$

which provides a Euclidean path integral description of vacuum tunneling phenomena in $A_0 = 0$ gauge.

Thus the validity of the Euclidean path integral description of the vacuum tunneling phenomena in $SU(2)$ Yang–Mills theory has been proved from the probability theoretical point of view.

4.2. Most Probable Tunneling Path and Instanton. We found the vacuum tunneling amplitude of the quantized Yang–Mills field in $A_0 = 0$ gauge be given by the Wiener integral (4.27). This can be written also in a functional path integral form (4.29). However we can no longer utilize equation (4.29) to derive a rigorous probability theoretical characterization of instantons since the functional path integral expression of the Wiener measure (4.28) has only a formal meaning.

Let us start with the transition probability law of the quantized Yang–Mills field $A_i^a(t)$, $-\infty < t < \infty$, in the vacuum state (4.26). It is convenient to approximate the Wiener integral in equation (4.26) by taking only an n-tuple functional integral account:

$$\int \delta[\mathbf{A}(s)-\mathbf{A}]^{*}\mu_{w}^{u,'\mathbf{A}}[\delta\mathbf{A}(\cdot)]\cdots \simeq \gamma\cdot \int \exp\left[-\frac{\|\mathbf{A}-\mathbf{A}_n\|^2}{2(s-t_n)\hbar}\right]\cdots$$

$$\times \exp\left[-\frac{\|\mathbf{A}_1-'\mathbf{A}\|^2}{2(t_1-u)\hbar}\right]\delta\mathbf{A}_n\cdots\delta\mathbf{A}_1\cdots \tag{4.31}$$

with $s>t_n>\cdots>t_1>u$ (Yasue, 1978e), where

$$\|\mathbf{A}\|=\left[\int A_i^a(x)A_i^a(x)d^3x\right]^{1/2}$$

is a norm on $^{*}\mathfrak{S}(\mathbb{R}^3)$ and $\gamma\in{}^{*}\mathbb{R}$ an infinitesimal constant. Then equation (4.26) becomes

$$Q[\mathbf{A};s|'\mathbf{A};u]\simeq\left[\frac{\Omega(\mathbf{A})}{\Omega('\mathbf{A})}\right]\cdot\gamma\int \exp\left[-\frac{\|\mathbf{A}-\mathbf{A}_n\|^2}{2(s-t_n)\hbar}\right]\cdots \exp\left[-\frac{\|\mathbf{A}_1-'\mathbf{A}\|^2}{2(t_1-u)\hbar}\right]$$

$$\times \exp\left\{-\frac{1}{2\hbar}\left[\left\|\mathbf{B}\left(\frac{\mathbf{A}+\mathbf{A}_n}{2}\right)\right\|^2\cdot(s-t_n)+\cdots\right.\right.$$

$$\left.\left.+\left\|\mathbf{B}\left(\frac{\mathbf{A}_1+'\mathbf{A}}{2}\right)\right\|^2\cdot(t_1-u)\right]\right\}\delta\mathbf{A}_n\cdots\delta\mathbf{A}_1$$

$$=\frac{\Omega(\mathbf{A})}{\Omega('\mathbf{A})}\int \exp\left\{-\frac{1}{2\hbar}\left[\left(\frac{\|\mathbf{A}-\mathbf{A}_n\|}{s-t_n}\right)^2+\left\|\mathbf{B}\left(\frac{\mathbf{A}+\mathbf{A}_n}{2}\right)\right\|^2\right]\cdot(s-t_n)\right.$$

$$\left.-\cdots-\frac{1}{2\hbar}\left[\left(\frac{\|\mathbf{A}_1-'\mathbf{A}\|}{t_1-u}\right)^2+\left\|\mathbf{B}\left(\frac{\mathbf{A}_1+'\mathbf{A}}{2}\right)\right\|^2\right]\cdot(t_1-u)\right\}$$

$$\delta\mathbf{A}_n\cdots\delta\mathbf{A}_1. \tag{4.32}$$

Now what is left for us is to replace each functional integration in equation (4.32) by taking the maximum value in the exponent, regarding the fact that the most probable value of a Gaussian distribution might dominate the Gaussian integral, obtaining

$$Q[\mathbf{A};s|'\mathbf{A};u]\simeq[\Omega(\mathbf{A})/\Omega('\mathbf{A})]$$
$$\cdot\gamma\exp\left\{-\frac{1}{2\hbar}\left[\left(\frac{\|\mathbf{A}-\mathbf{A}_n\|}{s-t_n}\right)^2+\left\|\mathbf{B}\left(\frac{\mathbf{A}+\mathbf{A}_n}{2}\right)\right\|^2\right]\cdot(s-t_n)-\cdots\right.$$
$$\left.-\frac{1}{2\hbar}\left[\left(\frac{\|\mathbf{A}_1-'\mathbf{A}\|}{t_1-u}\right)^2+\left\|\mathbf{B}\left(\frac{\mathbf{A}_1+'\mathbf{A}}{2}\right)\right\|^2\right]\cdot(t_1-u)\right\}_{\max} \tag{4.33}$$

where $[\cdot]_{\max}$ means to take a maximum value (Yasue, 1978e). Passing to the limit $n\to\infty$, we finally obtain an approximate expression of the transition probability law of the quantized vacuum field configuration $A_i^a(t)$, $-\infty<t<\infty$, as follows:

$$Q[\mathbf{A};s|'\mathbf{A};u]\simeq\left[\frac{\Omega(\mathbf{A})}{\Omega('\mathbf{A})}\right]\cdot\gamma\exp\left[-\frac{1}{\hbar}\int_u^s dt\int d^3x\left(\frac{1}{2}\dot{A}_i^a\dot{A}_i^a+\frac{1}{2}B_i^aB_i^a\right)\right]_{\max} \tag{4.34}$$

Correspondingly the vacuum tunneling amplitude (4.27) has an approximative expression

$$\mathrm{Amp}[{}^p\mathbf{A};\infty|{}^q\mathbf{A};-\infty]\simeq\gamma\cdot\exp\left[-\frac{1}{\hbar}\int_{-\infty}^{\infty}dt\int d^3x\left(\frac{1}{2}\dot{A}_i^a\dot{A}_i^a+\frac{1}{2}B_i^aB_i^a\right)\right]_{\max} \tag{4.35}$$

This is the well-known WKB prescription (Coleman, 1977).

Let us introduce a notion of the most probable tunneling path $\bar{A}_i^a(\mathbf{x},t)$. It is a classical Euclidean Yang–Mills field which minimizes the Euclidean action

$$S_E[\mathbf{A}]=\int\left(\frac{1}{2}\dot{A}_i^a\dot{A}_i^a+\frac{1}{2}B_i^aB_i^a\right)dt\,d^3x$$
$$=\frac{1}{4}\int\sum_{a=1}^{3}\sum_{\mu,\nu=0}^{3}F_{\mu\nu}^a F_{\mu\nu}^a\,d^4x \tag{4.36}$$

under the boundary conditions

$$\lim_{t\to\infty} \bar{A}_i^a(\mathbf{x},t) = {}^pA_i^a(\mathbf{x}) \tag{4.37}$$

$$\lim_{t\to-\infty} \bar{A}_i^a(\mathbf{x},t) = {}^qA_i^a(\mathbf{x}) \tag{4.38}$$

Then equation (4.35) becomes

$$\begin{aligned}\mathrm{Amp}[{}^p\mathbf{A};\infty|{}^q\mathbf{A};-\infty] &\simeq \gamma\cdot\exp\left[-\frac{1}{\hbar}\int\left(\frac{1}{2}\dot{\bar{A}}_i^a\dot{\bar{A}}_i^a + \frac{1}{2}\bar{B}_i^a\bar{B}_i^a\right)dt\,d^3x\right]\\ &= \gamma\cdot\exp\left(-\frac{1}{\hbar}S_E[\bar{A}]\right)\end{aligned} \tag{4.39}$$

This expression was known as "Onsager–Machlup formula" in non-equilibrium statistical physics (Yasue, 1978e).

Thus we have found that the instanton, associated with the Euclidean action-minimum classical field configuration, manifests the most probable tunneling path of the quantized Yang–Mills field between metastable vacuum states.

4.3 Quantum Decay Process of Metastable Vacuum States. To illustrate the mechanism of the vacuum tunneling more clearly, we shall investigate the quantum decay process of metastable vacuum states by solving the stochastic differential equation (4.17) explicitly. It is convenient to rewrite equation (4.17) in terms of the white noise (Hida, 1970, 1975b), obtaining

$$\dot{A}_i^a(\mathbf{x},t) = U_i^a[\mathbf{A}(\mathbf{x},t)] + Z_i^a(\mathbf{x},t) \tag{4.40}$$

where $Z_i^a(\mathbf{x},t) = \dot{W}_i^a(\mathbf{x},t)$ denotes a Gaussian white noise with mean 0 and covariance

$$\mathbb{E}\{Z_i^a(\mathbf{x},t)Z_j^b(\mathbf{y},u)\} = \hbar\delta^{ab}\delta_{ij}\delta(t-u)\delta^3(\mathbf{x}-\mathbf{y}) \tag{4.41}$$

This is simply because one can consider the problem of the quantum decay process in a concrete mathematical framework of distribution theory.

As the transformation $U_i^a(\cdot)$ in ${}^*\mathfrak{S}(\mathbb{R}^3)$ is related with the vacuum state wave functional $\Omega(\mathbf{A})$ by equation (4.18), first of all, we have to construct a physically relevant vacuum state which is invariant under gauge transformations. It is known to be a coherent superposition of Gaussian functionals peaked around each metastable vacuum state ${}^qA_i(x)$

(Callan et al., 1976; Jackiw and Rebbi, 1976; Jackiw, 1977). Such a gauge-invariant vacuum state wave functional is parametrized by an angle θ

$$\Omega_\theta(\mathbf{A}) = \sum_{q\in\mathbf{z}} e^{iq\theta}\Phi_\omega(\mathbf{A}-\mathbf{q}_A) \tag{4.42}$$

with

$$\Phi_\omega(\mathbf{A}) = \exp\left[-\frac{1}{2\hbar}\int A_i^a(x)\omega A_i^a(x)\,d^3x\right] \tag{4.43}$$

where ω is a positive linear operator chosen in a way that $\Phi_\omega(\mathbf{A}-{}^q\mathbf{A})$ is a local solution of the Schrödinger equation (4.10), that is, a solution to a harmonized Schrödinger equation around ${}^qA_i(\mathbf{x})$; $\omega=(-\partial_i^2)^{1/2}$. The gauge-invariant vacuum state of the quantized Yang–Mills field is a Bloch state (4.42).

To describe a quantum decay process of the metastable vacuum state ${}^qA_i(\mathbf{x})$, we shall approximate equation (4.42) by taking only a Gaussian functional peaked around ${}^qA_i(\mathbf{x})$ into account, obtaining

$$\Omega_\theta(\mathbf{A}) \simeq e^{iq\theta}\Phi_\omega(\mathbf{A}-{}^q\mathbf{A}) \tag{4.44}$$

Then equation (4.40) becomes

$$A_i^a(\mathbf{x},t) = \omega\left[A_i^a(\mathbf{x},t) - {}^qA_i^a(\mathbf{x})\right] + Z_i^a(\mathbf{x},t) \tag{4.45}$$

which is a linear inhomogeneous stochastic differential equation of white noise type (Yasue, 1978f). A solution of equation (4.45) under the initial condition $A_i^a(\mathbf{x},0) = {}^qA_i^a(\mathbf{x})$ manifests the quantum decay process of the metastable vacuum state ${}^q\mathbf{A}\in{}^*\mathfrak{S}(\mathbb{R}^3)$.

Such a solution to equation (4.45) can be obtained by introducing a contraction semigroup on ${}^*\mathfrak{S}(\mathbb{R}^3)$:

$$T(t) = \exp(-\omega t)\,\delta^{ab}\delta_{ij}, \qquad t\geqslant 0 \tag{4.46}$$

Namely,

$$A_i^a(\mathbf{x},t) = {}^qA_i^a(\mathbf{x}) + \int_0^t T(t-u)Z_i^a(\mathbf{x},u)\,du \tag{4.47}$$

solves equation (4.45) with the initial condition (Hida and Streit, 1977).

Equation (4.47) uniquely determines a distribution valued Gaussian process $A_i^a(\mathbf{x},t)$ with mean

$$E\{A_i^a(\mathbf{x},t)\} = {}^qA_i^a(\mathbf{x}) \tag{4.48}$$

and covariance

$$\mathbb{E}\left\{\int\left[A_i^a(x)-\mathbb{E}\{A_i^a(x)\}\right]f_i^a(x)d^4x\int\left[A_j^b(y)-\mathbb{E}\{A_j^b(y)\}\right]h_j^b(y)d^4y\right\}$$
$$=\int f_i^a(x)(-\Delta_4)^{-1}h_i^a(x)d^4x \tag{4.49}$$

where f_i^a and h_i^a belong to $\mathbb{S}(\mathbb{R}^4)$, $\Delta_4=\Sigma_{\mu=0}^4\partial_\mu^2$ is a four-dimensional Laplacian, and we have made an abbreviation $x=(\mathbf{x},t)$. This is nothing but a Euclidean–Markov field (Nelson, 1973) of Gaussian type (Yasue, 1978f).

Thus we have found, within the realm of the stochastic quantization, that the quantum decay process of the metastable vacuum state ${}^qA_i(\mathbf{x})$ can be represented by the Euclidean–Markov field (4.47). Needless to say, the integral that appears in the right-hand side of equation (4.49) is of infrared divergent nature. In other words, an object that manifests the quantum decay process of the metastable vacuum state has a long-range correlation such as the Coulomb gas. We may be allowed to consider the object as a quantum field theoretical version of the instanton which was originally introduced as an indication of the vacuum tunneling phenomena (Polyakov, 1975). The Euclidean–Markov field (4.47) may play an important role in quark confinement, as was suggested by Polyakov (1975).

4.4. Spatially Homogeneous σ Model. So far we have ignored couplings of fermions to the Yang–Mills field. In this section, to see the effect of the presence of fermion fields to the gauge field, we shall investigate a spatially homogeneous σ model from the probability theoretical point of view.

Consider a σ model described by a Lagrangean density

$$\mathfrak{L}=\tilde{\psi}\left[i\hbar\sum_{\mu=0}^{3}\gamma_\mu\partial_\mu-g\sigma\right]\psi+\frac{\hbar^2}{2}\sum_{\mu=0}^{3}{}'(\partial_\mu\sigma)^2-\frac{\lambda}{4}(\sigma^2-v^2)^2 \tag{4.50}$$

where g, λ, and v are constants and γ's Dirac matrices. The Dirac spinor ψ and the real scalar σ represent a fermion field and a boson field interacting with each other through the Yukawa coupling scheme (4.50). ($\sim$ denotes the covariant adjoint.) Field equations obtained from (4.50) are

$$\left(i\hbar \sum_{\mu=0}^{3} \gamma_\mu \partial_\mu - g\sigma\right)\psi = 0 \tag{4.51}$$

$$\hbar^2 \sum_{\mu=0}^{3}{}' \partial_\mu^2 \sigma + \lambda\sigma(\sigma^2 - v^2) = -g\tilde{\psi}\psi \tag{4.52}$$

We shall restrict ourselves to the spatially homogeneous case in which $\psi(\mathbf{x}, t) = \psi(t)$ and $\sigma(\mathbf{x}, t) = \sigma(t)$ hold. Then equations (4.51) and (4.52) become

$$i\hbar\dot{\psi}(t) = g\gamma_0 \sigma(t)\psi(t) \tag{4.53}$$

$$\ddot{\sigma}(t) = -\frac{\lambda}{\hbar^2}\sigma(t)\left[\sigma(t)^2 - v^2\right] - \frac{g}{\hbar^2}\tilde{\psi}(t)\psi(t) \tag{4.54}$$

Equation (4.53) can be solved in a product integral form (Nelson, 1969)

$$\psi(t) = \prod_{s=0}^{t} \left[1 - ig\gamma_0 \sigma(s)\, ds\right]\psi(0) \tag{4.55}$$

or a more familiar T-product form

$$\psi(t) = T\exp\left[-ig\gamma_0 \int_0^t \sigma(s)\, ds\right]\psi(0) \tag{4.56}$$

which yields a conservation law for the fermion number

$$\tilde{\psi}(t)\psi(t) = \tilde{\psi}(0)\psi(0) \tag{4.57}$$

Thus we find that the spatially homogeneous σ model is nothing but an anharmonic oscillator described by an equation of motion

$$\ddot{\sigma}(t) = -\frac{\lambda}{\hbar^2}\sigma(t)\left[\sigma(t)^2 - v^2\right] - \frac{g}{\hbar^2}\tilde{\psi}(0)\psi(0) \tag{4.58}$$

Quantum theoretical vacuum state of such an anharmonic oscillator as (4.58) is given by a wave function $u(\sigma) \in L_2(\mathbb{R})$ which satisfies the Schrödinger equation[2]

$$\left[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial\sigma^2} + \frac{\lambda}{4\hbar^2}(\sigma^2 - v^2)^2 + \frac{g}{\hbar^2}\tilde{\psi}(0)\psi(0)\sigma\right]u(\sigma) = Eu(\sigma) \tag{4.59}$$

[2]Jona-Lasinio (1978) investigated the vacuum tunneling of this type also in the framework of stochastic quantization.

The potential $V(\sigma)=\lambda(\sigma^2-v^2)^2/4\hbar^2+g\tilde{\psi}(0)\psi(0)\sigma/\hbar^2$ is bounded from below and has an absolute minimum $\sigma=-v$ and a relative minimum $\sigma=v$.

Now we shall calculate the decay rate of the metastable vacuum $\sigma=v$ within the realm of our probability theoretical formulation. As we have seen in Section 7, quantized motion of the vacuum field configuration is an (S3) process of the aforementioned type $\Sigma(t), -\infty<t<\infty$:

$$D\Sigma(t)=b\big(\Sigma(t)\big) \tag{4.60}$$

$$\Sigma(t)-\Sigma(t')=\int_{t'}^{t} b\big(\Sigma(s)\big)\,ds+W(t)-W(t') \tag{4.61}$$

where $b(\sigma)=\hbar\partial\log u(\sigma)/\partial\sigma$ and $W(t), -\infty<t<\infty$, is a Wiener process with diffusion constant $\hbar/2$. The (S3) process $\Sigma(t), -\infty<t<\infty$, has a stationary probability distribution $u(\sigma)^2$. Decay rate of the metastable vacuum state can be calculated by evaluating a transition probability law of the random process $\Sigma(t), -\infty<t<\infty$.

The transition probability density $p(\sigma,t|\sigma',t')$, with $t>t'$, of the process $\Sigma(t), -\infty<t<\infty$, is known to be an elementary solution of the Fokker–Planck equation

$$\frac{\partial}{\partial t}p=-\frac{\partial}{\partial\sigma}[b(\sigma)p]+\frac{\hbar}{2}\frac{\partial^2}{\partial\sigma^2}p \tag{4.62}$$

Introducing a relative transition probability density $f(\sigma,t|\sigma',t')$ by

$$p(\sigma,t|\sigma',u)=u(\sigma)f(\sigma,t|\sigma',t')u(\sigma')^{-1} \tag{4.63}$$

one can transform equation (4.62) into a self-adjoint form

$$-\hbar\frac{\partial}{\partial t}f=\left[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial\sigma^2}+V(\sigma)-E\right]f \tag{4.64}$$

The decay rate of the metastable vacuum state $\sigma=v$ is given by investigating an asymptotic behavior of the transition probability law $p(-v,t|v,0)$ for large t (Langer, 1967).

An elementary solution of equation (4.64) is given by a Wiener integral

$$f(\sigma,t|\sigma',t')=\exp\left[\frac{E(t-t')}{\hbar}\right]\int\exp\left[-\int_{t'}^{t}V(\xi(s))\,ds/\hbar\right]\delta[\xi(t)-\sigma]\,\mu_W^{t',\sigma'}(d\xi) \tag{4.65}$$

where $\mu_w^{t',\sigma'}$ denotes a Wiener measure with diffusion constant $\hbar/2$ and starting point (t',σ'). Thus the transition probability density of the (S3) process $\Sigma(t)$, $-\infty<t<\infty, p(-v,t|v,0)$ can be written as

$$p(-v,t|v,0)$$
$$=\frac{u(-v)}{u(v)}\cdot\exp(E\cdot t/\hbar)\int\exp\left[-\int_0^t V(\xi(s))\,ds/\hbar\right]\delta[\xi(t)+v]\,\mu_w^v(d\xi) \quad (4.66)$$

Now what is left for us is to compute the Wiener integral (4.66) and to obtain the tunneling probability. Let $\{u_n(\sigma)\}_{n=0}^{\infty}\subset L_2(\mathbb{R})[u_0(\sigma)=u(\sigma)$ and $E_0=E$, of course] be a C.N.O.S. of eigenfunctions of the Schrödinger equation

$$-\frac{\hbar^2}{2}u_n''(\sigma)+V(\sigma)u_n(\sigma)=E_n u_n(\sigma) \quad (4.67)$$

$n=0,1,\cdots$, where the prime means to take a derivative with respect to σ. Then a tunneling probability between metastable vacua $\sigma=v$ and $\sigma=-v$ becomes

$$p(-v,t|v,0)=\frac{u(-v)}{u(v)}\cdot\sum_{n=0}^{\infty}u_n(-v)u_n(v)\exp[-(E_n-E)t/\hbar] \quad (4.68)$$

Since equation (4.68) has an asymptotic expression for large t,

$$p(-v,t|v,0)\simeq u(-v)^2\left\{1+\frac{u_1(-v)u_1(v)}{u(-v)u(v)}\cdot\exp\left[-\frac{(E_1-E)t}{\hbar}\right]\right\} \quad (4.69)$$

the decay rate of the metastable vacuum state $\sigma=-v$ is found to be $(E_1-E)/\hbar$ (Gildener and Patrascioiu, 1977).

The level splitting, i.e., the decay rate, can be computed immediately with the use of the WKB prescription.

From a perturbation theoretical view point, the presence of the fermion term $g\tilde{\psi}\tilde{\psi}\sigma/\hbar^2$ does not cause any first-order effect to the decay rate $(E_1-E)/\hbar$, because the unperturbed potential $\lambda(\sigma^2-v^2)^2/4\hbar^2$ is symmetric and the fermion term skew symmetric.

5. EPILOGUE

We have made a long journey around the realm of the stochastic quantization. Since we have now no "vacation" left for us at all, we must stop our "sightseeing" without investigating two large parts of the subject matter of quantum theory: spin and relativity. To close this paper, we shall present a "guide book" to those parts for the readers.

During the course of this lecture (June, 1978), we received a couple of nice works due to Dohrn and Guerra (1977, 1978) in which the stochastic quantization was generalized to include quantum mechanics on Riemannian manifold. They made use of the notion of stochastic parallel displacement of tensors, introduced by Itô (1976), with a geodesic correction. Such a generalization may allow us to incorporate the spin freedoms into quantum mechanics from our probability theoretical point of view (Dankel, 1970; Caubet, 1976). Following Itô (1976), a covariant (S3) process on an n-dimensional Riemannian manifold $(\mathfrak{M},g)$ can be constructed with the use of a stochastic moving frame $\{E_a(t), -\infty<t<\infty\}_{a=1}^{n} \subset T_0^1\mathfrak{M}$ (tangent bundle of $\mathfrak{M}$) attached to the process $X(t), -\infty<t<\infty$. They are solutions to the covariant stochastic differential equations of Fisk–Stratonovich type

$$dB^i(t) = E_a^i(t)\circ dW_a(t)$$

$$dX^i(t) = b^i(X(t),t)\,dt + dB^i(t)$$

$$dE_a^i(t) = -\Gamma_{jk}^i(X(t))E_a^k(t)\circ dX^j(t) + \tfrac{1}{2}R_j^i(X(t))E_a^j(t)\,dt$$

where $\{W_a(t)\}_{a=0}^{n}, -\infty<t<\infty$, is a Wiener process in $\mathbb{R}^n$ with diffusion constant $\hbar/2$, $b(\cdot,t)\in T_0^1\mathfrak{M}$, Γ_{jk}^i's connection coefficients, and $R_j^i\in T_1^1\mathfrak{M}$ curvature tensor, respectively. The Fisk–Stratonovich product $\circ$ is defined

$$Z(t)\circ dX(t) = Z(t)\,dX(t) + \tfrac{1}{2}dZ(T)\,dX(t)$$

(Of course, this is an abbreviation for its stochastic integral form.)

A generalization to including relativistic quantum mechanics needs a construction of an (S3) process on a four-dimensional pseudo-Riemannian manifold [or on the Minkowski space $\mathbb{M}=(\mathbb{R}^4,(+---))$] with the use of a proper-time parameter τ. Since positive indefiniteness of the metric tensor does not allow us to introduce such a "commutative" stochastic moving frame as $E_a(t)$'s, we are apt to generalize it to "noncommutative" one such as the γ matrices (Yasue, 1977b). Relativistic extension was also obtained by Lehr and Park (1977) in utilizing the notion of elementary time intervals.

We conclude this long journey with the following speculations.

The stochastic quantization procedure seems to present a powerful technique in investigating the complicated structure of the gauge theory vacuum. Evidently equations (4.17) and (4.20), which manifest the vacuum structure, were of the same form as those of dynamical critical phenomena in nonequilibrium statistical mechanics. Namely, the transition probability law Q of the quantized vacuum field configuration plays a role of "order parameter." One may utilize equations (4.17) and (4.20) in analyzing the phase transition of the gauge theory vacuum such as the quark confinement.

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